THE GEOMETRY OF 2-CALIBRATED MANIFOLDS

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ABSTRACT. We define 2-calibrated structures, which are analogs of symplectic structures in odd dimensions. We show the existence of differential topological constructions compatible with the structure.

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CONTENTS

1. INTRODUCTION

In recent years there has been an enormous success in the study of symplectic manifolds using approximately holomorphic methods. These methods –introduced by S. Donaldson in 1996 [9]– amount to treating symplectic manifolds as generalizations of Kähler manifolds. It is convenient to think of a symplectic manifold –once a compatible almost complex structure J has been fixed– as a Kähler manifold (P, J, Ω) for which the integrability condition for J has been dropped.

Let M be any hypersurface of the Kähler manifold (P, J, Ω). M inherits on the one hand a codimension 1 distribution D defined as the field of J-complex hyperplanes in TM, with an integrable almost complex structure $J \colon D \to D$ (a CR structure), and on the other hand a closed 2-form $\omega := \Omega_{|M}$ which is positive

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on D (and hence maximally non-degenerate). A *2-calibrated structure* on M is the structure one gets when the integrability assumption on $J\colon D \to D$ is dropped.

The behavior of D in a CR manifold can be read through its Levi-form $\mathcal{L}$, and a number of interesting cases can be singled out (M is assumed to be oriented or equivalently D to be co-oriented).

(1) If $\mathcal{L}$ is strictly positive (resp. negative) we get a strictly pseudo-convex (resp. pseudo-concave) CR structure; its non-integrable counterpart (if we forget about J) is a contact structure.
(2) If $\mathcal{L} \equiv 0$ then D integrates into a codimension 1 foliation whose leaves inherit a Kähler structure. The non-integrable analogs of these Levi-flat manifolds are a class of regular Poisson manifolds that include mapping tori associated to symplectomorphisms and cosymplectic structures (defined by a closed 1-form α and a closed 2-form ω such that $\alpha \wedge \omega^n$ is a volume form). When $n = 1$ these analogs are nothing but smooth taut foliations.
(3) If $n = 1$ and $\mathcal{L} \geq 0$, what we have is a class of structures that include all taut confoliations (see section 3.5 in [13]).

Definition 1. *A* 2*-calibrated structure on* M^{2n+1} *is a pair* (D,ω)*, where* D *is a codimension* 1 *distribution and* ω *a closed* 2*-form no-where degenerate on* D*.*

We call the triple (M,D,ω) *a* 2*-calibrated manifold. We say that* ω *is positive on* D *or that* ω *dominates* D*. If* D *is integrable we speak of* 2*-calibrated foliations.*

(M,D,ω) *is said to be integral if* $[\omega/2\pi] \in H^2(M;\mathbb{R})$ *is in the image of the integer cohomology. If that is the case, the pre-quantum line bundle* (L,∇) *is the unique –up to isomorphism– hermitian line bundle with connection with Chern class* h *and curvature* $-i\omega$*, where* h *is some fixed integer lift of* $[\omega/2\pi]$*.*

Definition 2. W *is a 2-calibrated submanifold of* (M,D,ω) *if* $TW \cap D$ *has codimension* 1 *inside* TW *and it is dominated by* ω*. In other words,* W *must intersect* D *transversely and in a symplectic sub-distribution of* D*.*

The first application we will obtain is an analog of the existence of transversal cycles through any point of a taut foliation in M^3.

Proposition 1. *Let* (M^{2n+1},D,ω) *be a closed integral 2-calibrated manifold. For any fixed point* $y \in M$ *if* $k \in \mathbb{N}$ *is large enough, it is possible to find 2-calibrated submanifolds* W_k *of* M *of codimension* $2m$ *through* y *with the following property:*

- *The inclusion* $l\colon W_k \hookrightarrow M$ *induces maps* $l_*\colon \pi_j(W_k) \to \pi_j(M)$ *which are isomorphisms for* $j = 0,\ldots,n-m-1$ *and epimorphisms for* $j = n-m$*. The same result holds for the homology groups.*

If $m = 1$*, then the Poincaré dual of* $[W_k]$ *is* kh*.*

The above result is obtained by pulling back the $\mathbf{0}$ section of a vector bundle, and it extends the main result for contact manifolds of [20]. Something similar can be done with the determinantal loci of a homomorphism of complex vector bundles (see [28], theorem 1.6 and corollary 5.2 in [4]).

Proposition 2. *Let* (M,D,ω) *be a closed integral 2-calibrated manifold and* $L^{\otimes k}$ *the sequence of powers of the pre-quantum line bundle. Let* E*,* F *be hermitian vector bundles with connection and consider the sequence of bundles* $I_k = E^* \otimes F \otimes L^{\otimes k}$*. Then for all* k *large enough there exist sections* τ_k *of* I_k *for which the determinantal loci* $\Sigma^i(\tau_k) = \{x \in M |\ \mathrm{rank}(\tau_k(x)) = i\}$ *are integral 2-calibrated submanifolds stratifying* M*.*

The Poincarè Dual of the closure of $\Sigma^i(\tau_k)$ is given by the so called Porteous formula [31] (see also [33]):

$$\Delta_{E,F\otimes L^{\otimes k},i} = \begin{vmatrix} c_{n-i} & c_{n-i+1} & \cdots & \\ c_{n-i-1} & c_{n-i} & \cdots & \\ & & \ddots & \\ c_{n-m+1} & & \cdots & c_{n-i} \end{vmatrix},$$

where rank$E = m$, rank$F = n$ and c_j is the $j-th$ Chern class $c_j(F \otimes L^{\otimes k} - E)$ defined by the equality

$$1 + c_1(F \otimes L^{\otimes k} - E) + c_2(F \otimes L^{\otimes k} - E) + \cdots =$$
$$(1 + c_1(F \otimes L^{\otimes k}) + c_2(F \otimes L^{\otimes k}) + \cdots)/(1 + c_1(E) + c_2(E) + \cdots)$$

If the rank of E and F, and i are chosen so that $\Sigma^{i-1}(\tau_k)$ is empty, then $\Sigma^i(\tau_k)$ is a closed 2-calibrated submanifold.

Once we know that closed contact manifolds have contact submanifolds, it is an interesting problem to determine cohomology classes that can be realized by contact submanifolds.

Corollary 1. *Let (M, α), $\alpha \in \Omega^1(M)$, be a closed (exact) contact manifold of dimension $2n+1$. Let E, F be complex vector bundles and let i be a positive integer such that*

- *The codimension in $\mathrm{Hom}(E, F)$ of the strata of homomorphisms of rank i is not bigger than $2n + 1$.*
- *The codimension in $\mathrm{Hom}(E, F)$ of the strata of homomorphisms of rank $i - 1$ is bigger than $2n + 1$.*

Then there exist contact submanifolds whose Poincarè dual is $\Delta_{E,F,i}$.

The next application is an analog for 2-calibrated manifolds of the embedding theorem for symplectic manifolds of [28] (theorem 1.2), extending results of [29] for contact manifolds.

Corollary 2. *Let (M^{2n+1}, D, ω) be a closed integral 2-calibrated manifold. Then it is possible to find maps $\phi_k : M \to \mathbb{CP}^{2n}$ so that for all k large enough one has:*

- *ϕ_k is an immersion along D.*
- *$[\phi_k^* \omega_{FS}] = [k\omega]$, where ω_{FS} is the Fubini-Study 2-form of $\mathbb{CP}^{2n}$.*

In particular if (M^3, D) is a closed 3-manifold with a smooth taut confoliation, it is possible to find immersions along D in $\mathbb{CP}^2$.

The previous corollary can be improved in two directions:

Corollary 3. *(see [28], corollary 2.6) Let $(M^{2n+1}, \mathcal{D}, \omega)$ be a closed manifold with an integral 2-calibrated foliation. Then the maps of corollary 2 can be composed from the right with diffeomorphisms of M so that for all $k \gg 1$ the equality $[\phi_k^*\omega_{FS}] = [k\omega]$ holds also at the level of foliated 2-forms, i.e. $\phi_k^*\omega_{FS|\mathcal{D}} = kw_{|\mathcal{D}}$.*

The second improvement is that the immersion along D can be perturbed to be transversal to any finite collection of complex submanifolds of projective space.

Another application is the existence of Lefschetz pencil structures, introduced in [19].

Definition 3. *Let (M, D, ω) be a 2-calibrated manifold and $x \in M$. A chart $\varphi : (\mathbb{C}^n \times \mathbb{R}, 0) \to (M, x)$ is compatible with (D, ω) if at the origin it sends the foliation of $\mathbb{C}^n \times \mathbb{R}$ by complex hyperplanes into D, and $\varphi^*\omega(0)$ restricted to the subspace $\mathbb{C}^n \times \{0\}$ is of type $(1, 1)$.*

Definition 4. *(see [32]) A Lefschetz pencil structure for (M, D, ω) is a triple (f, B, Δ) where $B \subset M$ is a codimension 4 2-calibrated submanifold, and $f\colon M \backslash B \to \mathbb{CP}^1$ is a smooth map such that:*

1. *f is a submersion along D away from Δ, a 1-dimensional manifold transversal to D where the restriction of the differential of f to D vanishes.*
2. *Around any point $x \in \Delta$ there exist coordinates $z_1, \ldots, z_n, s$ compatible with (D, ω), and complex coordinates of $\mathbb{CP}^1$ such that*
$$f(z, s) = z_1^2 + \cdots + z_n^2 + t(s), \tag{1}$$
where $t \in C^\infty(\mathbb{R}, \mathbb{C})$.
3. *Around any point $x \in B$ there exist coordinates $z_1, \ldots, z_n, s$ compatible with (D, ω), and complex coordinates of $\mathbb{CP}^1$ such that $B \equiv z_1 = z_2 = 0$ and $f(z, s) = z_1/z_2$.*
4. *$f(\Delta)$ is an immersed curve.*

Theorem 1. *Let (M, D, ω) be an integral 2-calibrated foliation (M closed) and let h be an integer lift of $[\omega]$. Then for all $k \gg 1$ there exist Lefschetz pencils (f_k, B_k, Δ_k) such that:*

1. *The regular fibers are Poincarè dual to kh.*
2. *The inclusion $l\colon W_k \hookrightarrow M$ induces maps $l_*\colon \pi_j(W_k) \to \pi_j(M)$ (resp. $l_*\colon H_j(W_k; \mathbb{Z}) \to H_j(M; \mathbb{Z})$) which are isomorphism for $j \leq n-2$ and epimorphisms for $j = n-1$.*

All the stated results follow mostly from a more general principle of (estimated) transversality along D (theorems 2 and 3).

In a problem $\mathcal{P}$ of transversality *along* D we have three ingredients: (i) the bundle $E \to (M, D, \omega)$, (ii) the submanifold or more generally the stratification $\mathcal{S} \subset E$ and (iii) the section $\tau\colon M \to E$ to be perturbed to become transversal to $\mathcal{S}$.

In section 2 we will define the class of sections and bundles we will work with, the so called sequences of *very ample bundles* (definition 6) and *approximately holomorphic sections* (definition 7).

As in the approximately holomorphic theory for symplectic manifolds (see [9, 4]), transversality problems will be solved by patching local solutions. The right strategy to solve the corresponding local problem for sections is to turn them into local problems for approximately holomorphic functions. That will be done through the use of *reference sections*, which can be thought as the bump functions of the theory. The necessary local analysis needed to construct such sections is developed in section 3.

There is a second strategy to solve $\mathcal{P}$. It is not only that the natural example of a 2-calibrated structure is a hypersurface inside a symplectic manifold, but every 2-calibrated manifold (D co-oriented) admits a symplectization $(M \times [-\epsilon, \epsilon], \Omega)$ (lemma 4). We will introduce a new transversality problem $\bar{\mathcal{P}}$ for a stratification $\bar{\mathcal{S}}$ of a bundle $\bar{E} \to (M \times [-\epsilon, \epsilon], \Omega)$, so that a solution $\bar{\tau}\colon M \times [-\epsilon, \epsilon] \to \bar{E}$ to the new transversality problem $\bar{\mathcal{P}}$ restricts to $\bar{\tau}_{|M}$ a solution to $\mathcal{P}$. The advantage of this point of view is that since we are in a symplectic manifold, as long as the extension $\bar{\mathcal{P}}$ falls in the right class of problems we can use the existing approximately holomorphic theory for symplectic manifolds to solve it. Still, the existing approximately holomorphic theory turns out not to be enough for our purposes, so we need to develop further the *relative approximately holomorphic theory* introduced by J.P. Mohsen [27]. We will make an exposition of both the intrinsic and the relative approximately holomorphic theories, and we will prove the main transversality theorem using the latter.

In section 4 we give an account of the notion of estimated transversality of a section along a distribution. For the intrinsic theory (problem $\mathcal{P}$) the distribution

will be D, whereas for the relative theory the problem $\bar{\mathcal{P}}$ will amount to achieve transversality along $M \subset (M \times [-\epsilon, \epsilon], \Omega)$. We will also introduce the right class of stratifications $\mathcal{S}$ (already defined in the symplectic setting in [4]), the so called *approximately holomorphic finite Whitney stratifications*, whose strata roughly behave as the zero section of a vector bundle in the sense that locally they will be given by approximately holomorphic functions and they will be transversal enough to the fibers. The fundamental result (lemma 11) will be that locally estimated transversally along D (resp. M) of a section to $\mathcal{S}$ (resp. $\bar{\mathcal{S}}$) will be equivalent to estimated transversality along D (resp. M) to $\mathbf{0}$ of a related $\mathbb{C}^l$-valued approximately holomorphic function.

Section 5 is devoted to the study of bundles of pseudo-holomorphic jets, needed to obtain *generic* approximately holomorphic maps to projective spaces, constructed by projectivizing $(m+1)$-tuples of approximately holomorphic sections of powers of the pre-quantum line bundle $L^{\otimes k}$ (an analog of generic linear systems); genericity will be defined as the solution of a uniform *strong transversality* problem to a stratification $\mathcal{S}$ in these bundles of pseudo-holomorphic jets (definition 27). Several difficulties have to be overcome. Firstly, since we want to obtain a strong transversality result, the jet of the section to be perturbed has to be itself an approximately holomorphic section, so that the transversality problem falls in the right class, something which fails to hold due to the uniform positivity along D of the sequence $L^{\otimes k}$. This is solved by introducing a new connection in the bundles of pseudo-holomorphic jets (proposition 4). Secondly, we need to define an stratification $\mathcal{S}$ of the right kind (subsection 6.2). This is done in section 6 by introducing the bundles of pseudo-holomorphic jets for maps to projective spaces, and defining there $\mathbb{P}\mathcal{S}$ –a "linear" analog of the Thom-Boardman stratification–; $\mathcal{S}$ is constructed by pulling back $\mathbb{P}\mathcal{S}$ by the corresponding jet extension of the projectivization map $\pi\colon \mathbb{C}^{m+1}\backslash\{0\} \to \mathbb{CP}^m$. The properties of both the map and of $\mathbb{P}\mathcal{S}$ are used to conclude that $\mathcal{S}$ is indeed of the right kind, and thus the transversality problem falls in the right class (lemma 16). The necessary modifications for the relative theory are also described.

In section 7 we give the main strong transversality result.

The proofs of the theorems stated in this introduction are given in section 8.

Our results are based on the abundance of approximately holomorphic sections of very ample line bundles. In the integrable setting there are analogous results following two different strategies:

(i) In [14] E. Ghys gave conditions on a compact space laminated by Riemann surfaces for the existence of plenty of meromorphic functions. More generally, B. Deroin has extended those results to laminations by complex leaves without vanishing cycle, and endowed with positive hermitian line bundles [8]. The work of Ghys and Deroin proves the existence of leafwise holomorphic embeddings in projective spaces of the aforementioned laminated spaces (compare with corollary 2), though the maps –even in the case of smooth foliations– are in general only continuous in the transversal directions. The strategy they follow is working in the universal cover of the leaves of the lamination. Interestingly enough, Deroin's results are obtained by extending some techniques of approximately holomorphic geometry to the leaves, which are open complex manifolds with bounded geometry.

(ii) In [30] Ohsawa and Sibony gave a solution to the $\bar{\partial}$-Neumann problem with L^2-estimates for sections of a positive CR line bundle over a Levi-flat compact manifold. As a consequence they were able to produce CR embeddings in projective space of any prescribed order of regularity (though in general non-smooth).

Part of the results of the present paper were announced in [18, 19] (proposition 1, corollary 2, corollary 3, theorem 1 and theorem 2), where an account of the results available through an intrinsic approximately holomorphic theory was presented.

While a more detailed study of 2-calibrated structures is feasible, we do not think the results that could be obtained would be relevant enough to justify its undertaking.

There has been two main reasons to develop an approximately holomorphic theory for 2-calibrated structures:

The first one is because they contain contact structures and 2-calibrated foliations. Approximately holomorphic geometry has already been introduced in the contact setting [20, 32, 27, 29]. It has been used to construct compatible open book decompositions for contact manifolds of arbitrary dimension [16]. Our contribution in this paper to contact geometry is the construction of a large class of contact submanifolds and the determination of their homology class (corollary 1).

We want to propose 2-calibrated foliations as an interesting higher dimensional generalization of 3-dimensional taut foliations. In [23] it is shown that any such foliation $(M, \mathcal{D}, \omega)$ contains a 3-dimensional taut foliation $(W^3, \mathcal{D}_W) \hookrightarrow (M, \mathcal{D})$ so that the inclusion descends to a homeomorphism between leaf spaces. It is done by showing that W^3 can be chosen to intersect each leaf of $(M, \mathcal{D})$ in a unique connected component; this is somehow surprising since the leafs are in general open submanifolds which can be dense in M. The proof uses the leafwise symplectic parallel transport associated to Lefschetz pencil structures; it relies on the ideas of Seidel ([34], section 1).

The second reason to develop an approximately holomorphic theory for 2-calibrated structures is that sometimes they appear as auxiliary structures.

If M is an odd dimensional manifold and ω a maximally non-degenerate closed 2-form, any distribution D complementary to the kernel of ω endows M with a 2-calibrated structure. In [24] this idea was applied to almost contact manifolds to construct (via approximately holomorphic theory) open book decomposition with control on the topology of the leaves.

If $(M, D, J) \hookrightarrow (\mathbb{CP}^N, \omega_{FS})$ is a *projective CR manifold*, then it is possible to adapt the relative approximately holomorphic geometry to find sequences of pencils of hypersurfaces of large enough degree whose restriction to (M, D, J) defines a Lefschetz pencil structure, i.e. a CR morse function away from a CR submanifold of base points [22].

All the applications outlined so far for contact manifolds, 2-calibrated foliations and projective CR manifolds use at most pseudo-holomorphic 1-jets.

If a projective CR manifold is Levi-flat then it makes sense to speak about r-generic CR functions (defined to be leafwise r-generic holomorphic functions). It can be shown that for all $k \gg 1$ there exist *r-generic linear systems*: linear systems of $\mathcal{O}(k) \to \mathbb{CP}^N$ of rank $m(r)$ whose restriction to M define r-generic CR functions away from base points [22]. The existence of such functions can be easily stated as a transversality problem (along $\mathcal{D}$) $\mathcal{P}_{int}$ in the bundle of CR r-jets of CR maps from M to $\mathbb{CP}^{m(r)}$. One has to show that it can be *linearized* to a transversality problem $\mathcal{P}_{lin}$ (the bundle, the stratification and the notion of CR r-jet all have to be replaced by *linear* analogs) that fits into the ones solved in theorem 3 (necessarily using the relative theory and with the reference sections belonging to $\mathcal{O}(k)$). Then it has to be checked that the solution to $\mathcal{P}_{lin}$ gives rise to a solution of $\mathcal{P}_{int}$.

We think that the existence of r-generic linear systems for projective Levi-flat CR manifolds is a relevant result by itself and justifies the extension of the approximately holomorphic theory to higher order jets, which is technically awkward. We expect it to be useful to analyze such manifolds. For example one can use it to

define r-generic functions $f\colon (M^{2n+1}, \mathcal{D}, J) \to \mathbb{CP}^n$ (with no base points) for which the regular level sets are unions of circles (with variable number of components), and using the analysis of the singularities define a dynamical system transversal to $\mathcal{D}$ (at least for low values of $n \geq 2$); by iterating the Lefschetz pencil construction (the dimensional induction of [6], section 5) one can also define maps to $\mathbb{CP}^{n-1}$ whose fibers (by [23]) are 3-manifolds intersecting each leaf of $\mathcal{D}$ in a connected Riemann surface.

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2. Ample bundles and approximately holomorphic sections

Let (M, D, ω) be an integral 2-calibrated manifold. Let us fix once and for all a compatible almost complex structure $J\colon D \to D$, and a metric g so that $g_{|D} = \omega(\cdot, J)$. The kernel of ω is asked to be orthogonal to D, being the only reason that it makes some of the computations of the local theory simpler. Notice that for any such metric the closed $2n$-form ω^n is a calibration for D (see [17]).

If we forget about the 2-form what remains is the following structure.

Definition 5. *An almost CR structure is a tuple (M, D, J, g) where D is a codimension 1 distribution, $J\colon D \to D$ an almost complex structure and g a metric whose restriction to D is compatible with J (J is g-orthogonal and g-antisymmetric).*

Let $(L, \nabla) \to M$ be any hermitian line bundle (or more generally vector bundle) with compatible connection. Let us call $\hat{D}$ the pullback to L of D; let $\hat{J}$ and $\hat{g}$ be the almost complex structure and metric on L which extend the hermitian structure on the fibers and are defined on the horizontal distribution associated to ∇ by pulling back J and g respectively. Then $(L, \hat{D}, \hat{J}, \hat{g})$ is an almost CR manifold.

Our goal is being able to construct sections $\tau\colon M \to L$ which (i) are close enough to be *holomorphic* ($\tau_* J = \hat{J}\tau_*$) and (ii) transversal to suitable submanifolds of the total space of L (we prefer to use the adjective almost holomorphic instead of almost CR to be consistent with the terminology of [20] and [32]; we will just speak of CR sections when the base space is a Levi-flat CR manifold). That forces us to impose conditions on L, or actually on the curvature of the connection ∇.

To determine the right condition we go back for a moment to the symplectic theory, or more generally almost complex theory [4], where it is known that the right bundles are the so called *ample bundles*. Roughly speaking one demands the curvature to be of type $(1,1)$ –so that at small scale the induced almost complex structure on the total space is nearly integrable– and positive, which implies the existence of plenty of nearly holomorphic sections.

Definition 6. *(see [4], definition 2.1) Given c, δ positive real numbers, a hermitian line bundle with compatible connection $(L, \nabla) \to (M, D, J, g)$ is (c, δ)-D-ample (or simply ample) if its curvature F verifies $iF(v, Jv) \geq cg(v, v), \forall v \in D$ (and hence it is non-degenerate) and $|F_{|D} - F^{1,1}_{|D}|_g \leq \delta$, where we use the supremum norm.*

A sequence of hermitian line bundles with compatible connections (L_k, ∇_k) is asymptotically very ample (or just very ample) if fixed positive constants δ, $(C_j)_{j \geq 0}$ and a sequence $c_k \to \infty$ exist, so that from some $k_0 \in \mathbb{N}$ on the following inequalities for the curvatures F_k hold:

(1) $iF_k(v, Jv) \geq c_k g(v, v), \forall v \in D$
(2) $|F_{k|D} - F_{k|D}^{1,1}|_g \leq \delta c_k^{1/2}$
(3) $|\nabla^j F_k|_g \leq C_j c_k$

Another motivation for the previous definition is the case of Levi-flat CR manifolds, where according to the results of Ohsawa and Sibony [30] leafwise positivity grants the existence of plenty of CR sections.

The pre-quantum line bundle L of an integral 2-calibrated manifold (M, D, ω) is ample, and its tensor powers $L^{\otimes k}$ define a very ample sequence of line bundles (with $c_k = k$, $\delta = 0$).

From now on we will only consider almost CR structures on 2-calibrated manifolds defined by compatible almost complex structures and metrics. Similarly, we will only consider the sequence $L^{\otimes k}$. Anyhow all the results stated in this section remain valid for any very ample sequence of line bundles L_k over an almost CR manifold, by just putting L_k, iF_k, c_k instead of $L^{\otimes k}, k\omega, k$. Notice however that for the sequence of symplectic forms iF_k, their kernels vary with k (and in particular the are not all perpendicular to D, as it is the case for $k\omega$).

If $\tau_k \in \Gamma(L^{\otimes k})$, using J the restriction of $\nabla\tau_k$ to D can be written

$$\nabla_D \tau_k = \partial\tau_k + \bar{\partial}\tau_k,\ \partial\tau_k \in \Gamma(D^{*1,0} \otimes L^{\otimes k}),\ \bar{\partial}\tau_k \in \Gamma(D^{*0,1} \otimes L^{\otimes k})$$

We can see $\bar{\partial}\tau_k$ as a section of $T^*M \otimes L^{\otimes k}$ by declaring it to vanish on $D^\perp$, and then use the Levi-Civita connection on T^*M to define $\nabla^{r-1}\bar{\partial}\tau_k \in \Gamma(T^*M^{\otimes r} \otimes L^{\otimes k})$.

Let us denote the rescaled metric kg by g_k.

Definition 7. *A sequence of sections τ_k of $L^{\otimes k}$ is approximately J-holomorphic (or approximately holomorphic or simply A.H.) if positive constants $(C_j)_{j\geq 0}$ exist such that:*

$$|\nabla^j \tau_k|_{g_k} \leq C_j,\ \ |\nabla^{j-1}\bar{\partial}\tau_k|_{g_k} \leq C_j k^{-1/2},$$

Remark 1. *The original notion of A.H. sequence introduced in* [20, 32] *is a bit more general than definition 7. The difference –as well as the fact that only a finite number of derivatives were taken into account– is that the direction orthogonal to D had a different treatment. The main theorem of* [20] *produced appropriate A.H. sequences of sections with good control on any finite number of derivatives along D, but little along $D^\perp$. Using the relative theory one can obtain "better" solutions (with control in all directions), so we do not need to use the technically more complicated definition of* [20, 32].

3. The local A.H. theory

Maybe the most important idea on Donaldson's work [9] was the construction of localized A.H. sections (inspired in the work of Tian [35]) by adopting a unitary point of view instead of a holomorphic one. The use of a unitary connection in a Darboux chart allowed him to find a model coupled Cauchy-Riemann equation invariant under rescaling –provided one worked in the appropriate tensor power of the pre-quantum line bundle– and explicitly write *concentrated solutions* giving rise to *reference sections*.

The local theory, both using an intrinsic construction or the symplectization to be introduced in subsection 3.1, is based on the choice of appropriate families of charts. In the intrinsic local theory we need as well a local model for the coupled Cauchy-Riemann equations and a good choice of explicit solution. But both the model and the solution are quite easy to determine (after Donaldson's work).

The local model for the intrinsic approximately holomorphic theory in 2-calibrated manifolds, that can only be achieved asymptotically when $k \to \infty$, is the following:

the domain is $\mathbb{C}^n \times \mathbb{R}$, with coordinates $z^1, \dots, z^n, s$ (sometimes we write them as $x^1, \dots, x^{2n+1}$ or $x^1, \dots, x^{2n}, s$). The distribution D_h is given by the level hyperplanes of the vertical or real coordinate s. The identification of each leaf with $\mathbb{C}^n$ means that we have fixed the leafwise canonical almost complex structure J_0. The metric is the Euclidean one g_0 with Levi-Civita connection d (usual partial derivatives), and the distance is the Euclidean norm $|\cdot|$.

The last element in the base space is the 2-form, which is required to be

$$\omega_0 = \frac{i}{2}\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i \tag{2}$$

We ask for a choice of unitary trivialization of the line bundle whose connection form is

$$A = \frac{1}{4}\sum_{i=1}^{n} z^i d\bar{z}^i - \bar{z}^i dz^i \tag{3}$$

In $\mathbb{R}^N$ with coordinates $x^1, \dots, x^N$ let $\mathbb{R}^p$ denote the distribution by p-planes span $< \partial/\partial x^{i_1}, \dots, \partial/\partial x^{i_p} >$, $1 \leq i_1 < \cdots < i_p \leq N$; its Euclidean orthogonal is denoted by $\mathbb{R}^{N-p}$. If we have a distribution D' of dimension p in $\mathbb{R}^N$ which is transversal to $\mathbb{R}^{N-p}$, we can measure its distance to $\mathbb{R}^p$ as follows: let v^{i_l}, $l = 1, \dots, p$, be the vector field in $\mathbb{R}^{N-p}$ such that $\partial/\partial x^{i_l} + v^{i_l} \in D'$. Then define

$$|\mathrm{d}^j(\mathbb{R}^p - D')|_{g_0} = \max\{|\mathrm{d}^j v^{i_1}|_{g_0}, \dots, |\mathrm{d}^j v^{i_p}|_{g_0}\}$$

In the previous local model let us denote the line field spanned by $\partial/\partial s$ by D_v. According to the previous paragraph we can measure the distance in $\mathbb{C}^n \times \mathbb{R}$ to D_h (resp. D_v) of any codimension 1 (resp. dimension 1) distribution transversal to D_v (resp. D_h).

Definition 8. *Let $\varphi_{k,x} \colon (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$, for all $x \in M$ and all $k \gg 1$, be a family of charts with coordinates $z_k^1, \dots, z_k^n, s_k$. We call them a family of approximately holomorphic coordinates if there exist constants independent of k, x (uniform) so that the following estimates hold for all $k \gg 1$ in the points of $B(0, O(k^{1/2}))$:*

(1) *The Euclidean and the induced metric are comparable to any order, i.e.*

$$\frac{1}{\gamma} g_0 \leq g_k \leq \gamma g_0,\ \gamma > 0, \text{and } |\nabla^j \varphi_{k,x}^{-1}|_{g_0} \leq O(k^{-1/2}), \forall j \geq 2$$

(2) *The kernel of ω, which is $D^\perp$, is sent to a line field $\varphi_{k,x}^* D^\perp$ transversal to D_h and such that*

$$\begin{aligned} |\varphi_{k,x}^* D^\perp - D_v|_{g_0} &\leq |(z_k, s_k)| O(k^{-1/2}), \\ |\nabla^j(\varphi_{k,x}^* D^\perp - D_v)|_{g_0} &\leq O(k^{-1/2}),\ \forall j \geq 1 \end{aligned}$$

The pullback of D is transversal to D_v and

$$\begin{aligned} |\varphi_{k,x}^* D - D_h|_{g_0} &\leq |(z_k, s_k)| O(k^{-1/2}), \\ |\nabla^j(\varphi_{k,x}^* D - D_h)|_{g_0} &\leq O(k^{-1/2}),\ \forall j \geq 1 \end{aligned}$$

(3) *Regarding the antiholomorphic components,*

$$\begin{aligned} |\bar{\partial}\varphi_{k,x}^{-1}(z_k, s_k)|_{g_0} &\leq (|(z_k, s_k)|) O(k^{-1/2}), \\ |\nabla^j \bar{\partial}\varphi_{k,x}^{-1}(z_k, s_k)|_{g_0} &\leq O(k^{-1/2}), \forall j \geq 1, \end{aligned}$$

where $\bar{\partial}\varphi_{k,x}^{-1}$ is the antiholomorphic component of $\nabla_D(\pi_{D_h} \circ \varphi_{k,x}^{-1})$, with $\pi_{D_h} \colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n$ the projection onto the first factor.

We speak of Darboux coordinates when the additional condition $\varphi_{k,x}^* k\omega = \omega_0$ holds.

Remark 2. *According to condition 2 (resp. 3) we have* $D \oplus D^{\perp} = D_h \oplus D_v$ *(resp.* $J = J_0$*) at the origin. For most of our constructions it is enough to require the equality up to a summand of size* $O(k^{-1/2})$ *at most, but since these conditions are needed to prove results concerning pseudo-holomorphic jets (in particular the identities concerning local representations and subsets of transversal holonomy of lemma 14) we choose to ask for them from the very beginning.*

Remark 3. *If we are in an almost complex manifold then conditions 1 and 3 (2 makes no sense) recover the notion of approximately holomorphic charts (resp. Darboux charts if we add the Darboux condition on the 2-form).*

Lemma 1. *Let* (M, D, ω) *a compact 2-calibrated manifold (with* J, g *already fixed). Then a family of Darboux charts can always be constructed.*

Proof. Let us fix a family of charts $\psi_x \colon \mathbb{R}^{2n+1} \to U_x$ depending smoothly on x chosen on a small subset M_1 of M, so that $D \oplus D^{\perp} = D_h \oplus D_v$ at the origin. Denote by $x^1, \ldots, x^{2n}, s$ the coordinates on $\mathbb{R}^{2n+1}$. We compose ψ_x with the diffeomorphism $\Theta_x \colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ which is the identity on $\mathbb{R}^{2n} \times \{0\}$, preserves setwise the horizontal foliation D_h and sends $\ker \psi_x^* \omega$ to D_v. The diffeomorphisms Θ_x depend smoothly on x.

Now we fix J_0 to identify $\mathbb{R}^{2n+1}$ with $\mathbb{C}^n \times \mathbb{R}$ and compose with an element of $\mathrm{Gl}(2n, \mathbb{R}) \subset \mathrm{Gl}(2n+1, \mathbb{R})$ (again depending smoothly on $x \in M_1$) so that we obtain charts φ_x for which the pullback of J at the origin equals J_0.

By compactness, we can cover M with a finite number of subsets $M_1, \ldots, M_h$ in which the above charts can be constructed. In this way we obtain charts centered at every $x \in M$ (we might have more than one chart for each $x \in M$, but that is not relevant) so that the bounds on tensors pulled back from M to the domain of the charts will not depend on x .

We define $\varphi_{k,x}$ to be the composition $\varphi_x \circ \rho_{k^{-1/2}}$, where $\rho_{k^{-1/2}} \colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n \times \mathbb{R}$ is the dilatation by factor $k^{-1/2}$.

The equalities at the origin together with the smooth dependence on x of the steps taken before rescaling easily imply that we have obtained approximately holomorphic coordinates.

To obtain Darboux charts we need to modify $\varphi_{k,x}$ as follows: we apply Darboux' lemma with estimates (lemma 2.2 in [4]) to the almost complex manifolds $(\mathbb{C}^n \times \{0\}, \varphi_{k,x}^* J_{|\mathbb{C}^n \times \{0\}}, \varphi_{k,x}^* g_{|\mathbb{C}^n \times \{0\}})$ and the 2-forms $\varphi_{k,x}^* \omega_{|\mathbb{C}^n \times \{0\}}$. We get diffeomorphisms $\Psi_{k,x}$ on this leaf that are extended to $\mathbb{C}^n \times \mathbb{R}$ independently of the vertical coordinate s_k. The bounds on $\Psi_{k,x}$ and their derivatives coming from lemma 2.2 in [4] imply that the compositions $\varphi_{k,x} \circ \Psi_{k,x} \colon (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$ still define approximately holomorphic coordinates. Moreover, we can assume $J = J_0$ at the origin.

Observe that by construction the pullback of ω restricted to $\mathbb{C}^n \times \{0\}$ is $\omega_{0|\mathbb{C}^n \times \{0\}}$. Since $\partial/\partial s_k$ generates the kernel of $(\varphi_{k,x} \circ \Psi_{k,x})^* \omega$, the 2-form written in the basis associated to the coordinates $x_k^1, \ldots, x_k^{2n}, s_k$ contains no summand of the form $\omega_j dx_k^j \wedge ds_k$, $j = 1, \ldots, 2n$. Hence, it is characterized by is restriction to the leaves of D_h. But this restriction is $\omega_{0|\mathbb{C}^n \times \{s_k\}}$, for the translation along the vertical coordinate is a symplectomorphism.

Thus, ω is sent to ω_0 $\square$

Darboux charts are useful because there local computations become simpler.

Let d_k denote the distance defined by the metric g_k.

Recall that in the domain of a Darboux chart we can always fix $\xi_{k,x}$ a unitary trivialization of $L^{\otimes k}$ whose connection form is A (equation 3).

Lemma 2. *Let $\varphi_{k,x}\colon (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$ be a family of Darboux charts with coordinates $x_k^1, \dots, x_k^{2n}, x_k^{2n+1}$. Let F be a bundle associated to either TM or D and let $F_{k,x} \to B(0, O(k^{1/2})) \subset \mathbb{C}^n \times \mathbb{R}$ denote the pullback of $F \otimes L^{\otimes k}$ by $\varphi_{k,x}$. Associated to the Darboux coordinates there is a canonical trivialization $\zeta_{k,x,j}$ of $F_{k,x}$. Let T_k be a sequence of sections of $F \otimes L^{\otimes k}$ and use the frames $\zeta_{k,x,j} \otimes \xi_{k,x}$ to write $\varphi_{k,x}^* T_k$ locally as a function $T'_{k,x}$. Let P_j be a polynomial such that for any multi-index α of length j, $j = 0, \dots, r$, in the points of $B(0, O(k^{1/2}))$ and for all k large enough we have:*

$$\left| \frac{\partial}{\partial x_k^\alpha} T'_{k,x} \right|_{g_0} \le P_j(|(z_k, s_k)|) O(k^{-1/2})$$

Then $|\nabla^r T_k(y)|_{g_k} \le Q_r(d_k(x,y)) O(k^{-1/2})$, where the polynomial Q_r depends only on $P_1, \dots, P_r$. Conversely, from bounds using the global metric elements g_k, d_k, ∇ we obtain similar bounds for the local Euclidean elements.

Proof. A simple calculation based on points 1 and 2 and in the Darboux condition in definition 8. Also notice that the presence of the connection form and its derivatives is absorbed by the polynomial, since $|A| \le O(|(z_k, s_k)|)$ and its derivatives are of order $O(1)$. □

Lemma 2 admits different modifications. It holds in a similar fashion for bounds of order $O(1)$ instead of order $O(k^{-1/2})$ and also for sections T_k of F (with $F_{k,x}$ locally trivialized by $\zeta_{k,x,j}$).

Let us denote the $(0,1)$-component with respect to $J_0\colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n \times \mathbb{R}$ of the leafwise derivation operator d_{D_h} by $\bar\partial_0$.

Lemma 3. *Let $\varphi_{k,x}\colon (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$ be a family of Darboux charts with coordinates $x_k^1, \dots, x_k^{2n}, s_k$. Let $L_{k,x} \to B(0, O(k^{1/2})) \subset \mathbb{C}^n \times \mathbb{R}$ denote the pullback of $L^{\otimes k}$ by $\varphi_{k,x}$. Let τ_k be a sequence of sections of $L^{\otimes k}$ such that $\varphi_{k,x}^* \tau_k = f_{k,x} \xi_{k,x}$. Let P_j be a polynomial such that for any multiindices α, β of length j, $j = 0, \dots, r-1$ and j', $j' = 0, \dots, r$ respectively, in the points of $B(0, O(k^{1/2}))$ and for all k large enough the following inequalities hold:*

$$\left| \frac{\partial}{\partial x_k^\beta} f_{k,x} \right|_{g_0} \le P'_j(|(z_k, s_k)|) O(1) \tag{4}$$

$$\left| \frac{\partial}{\partial x_k^\alpha} (\bar\partial_0 + A^{0,1}) f_{k,x} \right|_{g_0} \le P_j(|(z_k, s_k)|) O(k^{-1/2}) \tag{5}$$

Then

$$|\nabla^r \tau_k(y)|_{g_k} \le Q'_r(d_k(x,y)) O(1), \tag{6}$$

$$|\nabla^{r-1} \bar\partial \tau_k(y)|_{g_k} \le Q_{r-1}(d_k(x,y)) O(k^{-1/2}), \tag{7}$$

where the polynomial Q_{r-1} (resp. Q'_r) depends only on $P_1, \dots, P_{r-1}, P'_1, \dots, P'_r$ (resp. $P'_1, \dots, P'_r$). Conversely, from bounds using the global elements g_k, d_k, ∇, J we obtain similar bounds for $g_0, |\cdot|, \mathrm{d} + A, J_0$.

Proof. The equivalence between equations 4 and 6 is the content of lemma 2 for bounds of order O(1). The equivalence of equations 4, 5 and equations 6, 7 follows again easily from the properties of Darboux charts. We sketch the case $r = 1$.

From now on $\varphi_{k,x}^* J, \varphi_{k,x}^* D, \varphi_{k,x}^* g_k$ and all the tensors and sections pulled back to the domain of the charts will be simply denoted by $J, D, g_k, \dots$ whenever there is no risk of confusion.

Let e_i be any of the local vector fields associated to the first $2n$ coordinates. By condition 2 in definition 8 there exists u_i a local vector field such that $e_i + u_i$ is tangent to D and

$$|u_i|_{g_0} \leq |(z_k, s_k)|O(k^{-1/2}),\ |\mathrm{d}^j u_i|_{g_0} \leq O(k^{-1/2}),\ j \geq 1 \tag{8}$$

The endomorphism J is defined on D. We can use the orthogonal projection w.r.t g_0 onto D_h to induce out of J another almost complex structure $J_{D_h} \colon D_h \to D_h$.

Condition 3 in definition 8 implies that

$$|J_0 - J_{D_h}|_{g_0} \leq (|(z_k, s_k)|)O(k^{-1/2}),\ |\mathrm{d}^j(J_0 - J_{D_h})|_{g_0} \leq O(k^{-1/2}),\ j \geq 1 \tag{9}$$

By definition $\bar{\partial}_{e_i+u_i}\tau_k = 1/2\nabla_{e_i+u_i}\tau_k + i/2\nabla_{J(e_i+u_i)}\tau_k$.

Equation 8 combined with lemma 2 implies

$$|\nabla_{u_i}\tau_k|_{g_k} \leq P_1'(d_k(x,y))O(k^{-1/2})$$

Again equations 8 and 6, condition 3 in definition 8 and lemma 2 imply

$$|\nabla_{J(e_i+u_i)}\tau_k - \nabla_{J_h e_i}\tau_k|_{g_k} \leq P_1''(d_k(x,y))O(k^{-1/2})$$

Therefore, the bounds in equation 7 we want for $\bar{\partial}_{e_i+u_i}\tau_k$, are equivalent to the same kind of bounds for

$$1/2\nabla_{e_i}\tau_k + i/2\nabla_{J_h e_i}\tau_k,$$

and by equation 9 for

$$1/2\nabla_{e_i}\tau_k + i/2\nabla_{J_0 e_i}\tau_k$$

And by definition

$$1/2\nabla_{e_i}\tau_k + i/2\nabla_{J_0 e_i}\tau_k = ((\bar{\partial}_0 + A^{0,1})_{e_i} f_{k,x})\xi_{k,x}$$

Bounds for higher order derivatives are proven similarly. □

Definition 9. *(see [4], definition 2.2) A sequence of sections of $L^{\otimes k}$ has gaussian decay w.r.t. x if polynomials $(P_j)_{j\geq 0}$ and a constant $\lambda > 0$ exist so that $\forall y \in M$ and $\forall j \geq 0$,*

$$|\nabla^j \tau_k(y)|_{g_k} \leq P_j(d_k(x,y))\mathrm{e}^{-\lambda d_k(x,y)^2}$$

The main purpose of the use of Darboux charts is the construction of *reference sections* $\tau_{k,x}^{\mathrm{ref}}$.

Corollary 4. *Let (M, D, ω) be a compact 2-calibrated manifold. Then for all $x \in M$ A.H. sections $\tau_{k,x}^{\mathrm{ref}}$ with gaussian decay w.r.t. x can be constructed. The bounds are uniform on k, x and these sections have norm greater than some constant κ in $B_{g_k}(x, \rho)$, where $\kappa, \rho > 0$ are uniform on k, x.*

Proof. We follow Donaldson's ideas in [9], section 2. Let us fix Darboux charts and $\xi_{k,x}$ trivializations of $L^{\otimes k}$ for which the connection form is A. Let β be a standard cut-off function of a single variable, with $\beta(t) = 1$ when $|t| \leq 1/2$ and $\beta(t) = 0$ when $|t| \geq 1$.

Define $\beta_k(z_k, s_k) = \beta(k^{-1/6}|(z_k, s_k)|)$.

In the points where the derivatives of β_k do not vanish we have $|(z_k, s_k)| \geq O(k^{1/6})$. Using this inequality we deduce

$$\begin{aligned} |\mathrm{d}\beta_k|_{g_0} &\leq |(z_k, s_k)|^2 O(k^{-1/2}),\ |\mathrm{d}^2\beta_k|_{g_0} \leq |(z_k, s_k)|O(k^{-1/2}), \\ |\mathrm{d}^j\beta_k|_{g_0} &\leq O(k^{-1/2}),\ j \geq 3 \end{aligned} \tag{10}$$

Consider the function $f(z_k, s_k) = \mathrm{e}^{-|(z_k,s_k)|^2/4}$. We have

$$\bar{\partial}_0 f + A^{0,1} f = 0 \tag{11}$$

The reference sections are

$$\tau_{k,x}^{\mathrm{ref}} := \beta_k f \xi_{k,x} \tag{12}$$

Equation 10 implies that for any multi-index α of length $j \leq r$,

$$\left| \frac{\partial}{\partial x^\alpha} \beta_k f \right|_{g_0} \leq P_j(|(z_k, s_k)|)|f|O(1)$$

When we compute $\nabla^r \tau_{k,x}^{\mathrm{ref}}$ we can always factor out the function f. Therefore, a slight modification of lemma 2 gives the gaussian decay w.r.t. x:

$$|\nabla^r \tau_{k,x}^{\mathrm{ref}}(y)|_{g_k} \leq Q_r(d_k(x,y))\mathrm{e}^{-\lambda d_k(x,y)^2} O(1),\ \lambda > 0$$

The gaussian decay also implies

$$|\nabla^r \tau_{k,x}^{\mathrm{ref}}|_{g_k} \leq O(1)$$

The bound for $|\nabla^{r-1}\bar{\partial}\tau_{k,x}^{\mathrm{ref}}|_{g_k}$ is obtained using the same ideas: from equations 10 and 11 it follows that for any multi-index α of length $j \leq r-1$

$$\left| \frac{\partial}{\partial x^\alpha} (\bar{\partial}_0 + A^{0,1})\beta_k f \right|_{g_0} \leq P_j(|(z_k, s_k)|)|f|O(k^{-1/2})$$

Lemma 3 suitably modified gives for some $\lambda > 0$

$$|\nabla^{r-1}\bar{\partial}\tau_{k,x}^{\mathrm{ref}}|_{g_k} \leq Q_r(d_k(x,y))\mathrm{e}^{-\lambda d_k(x,y)^2} O(k^{-1/2}) \leq O(k^{-1/2})$$

The existence of constants $\kappa, \rho > 0$ such that $|\tau_{k,x}^{\mathrm{ref}}| \geq \kappa$ in $B_{g_k}(x,\rho)$, can be easily checked. □

We observe that many of the inequalities we are using (for global tensors) have the same pattern. We will introduce a definition that will avoid the excessive appearance in the notation of such inequalities.

Let E be a hermitian bundle with connection, F a bundle associated either to TM or to D and let E_k denote the sequence $F \otimes E \otimes L^{\otimes k}$.

Definition 10. *Let $T_{k,\lambda}$, $\lambda \in \Lambda$, be a sequence of sections of E_k. We say that $T_{k,\lambda}$ is C^r-approximately vanishing (or that the sequence vanishes in the C^r-approximate sense) and denote it by $T_{k,\lambda} \cong_r 0$, if positive constants $C_0, \ldots, C_r$ exist so that*

$$|\nabla^j T_{k,\lambda}|_{g_k} \leq C_j k^{-1/2},\ j = 0, \ldots, r \tag{13}$$

In what follows the parameter space Λ will amount either to one element (so we have a sequence of sections) or to the points of M.

Using the above language one of the conditions for a sequence τ_k of $L^{\otimes k}$ to be A.H. (definition 7) is that $\bar{\partial}\tau_k \in \Gamma(D^{*0,1} \otimes L^{\otimes k})$ has to be approximately vanishing.

Remark 4. *One final observation is that given τ_k an approximately holomorphic sequence of sections of $L^{\otimes k}$, we have defined $\nabla^{r-1}\bar{\partial}\tau_k \in T^*M^{\otimes r} \otimes L^{\otimes k}$ by taking covariant derivatives of $\bar{\partial}\tau_k$ thought as a section of $T^*M \otimes L^{\otimes k}$. We might have equally defined $\nabla^{r-1}\bar{\partial}\tau_k$ as the image of $\nabla^r\tau_k$ by the projection $\bar{p}_r \colon T^*M^{\otimes r} \otimes L^{\otimes k} \to T^*M^{\otimes r-1} \otimes D^{*0,1} \otimes L^{\otimes k}$, for using Darboux charts and lemmas 2 and 3 (with the inequalities $|\nabla^j \tau_k|_{g_k} \leq O(1)$, $j \geq 0$), one checks that $\bar{\partial}\tau_k \cong 0$ if and only if $|\bar{p}_j(\nabla^j\tau_k)|_{g_k} \leq O(k^{-1/2})$, $j \geq 1$.*

3.1. **Relative approximately holomorphic theory and symplectizations.**

Definition 11. *Let (P,Ω) be a symplectic manifold and (M,D,ω) a 2-calibrated manifold. We say that $l\colon (M,D,\omega)\hookrightarrow (P,\Omega)$ embeds M as a 2-calibrated submanifold of P if $l^*\Omega=\omega$.*

Lemma 4. *Let (M,D,ω) a compact co-oriented 2-calibrated manifold. Then it is possible to define a symplectization so that (M,D,ω) embeds as 2-calibrated submanifold. Any fixed compatible almost complex structure and metric can be extended to a compatible almost complex structure and metric in the symplectization.*

Proof. Let J and g be fixed c.a.c.s. and metric. The symplectization $(M\times[-\epsilon,\epsilon],J,g,\Omega)$ is constructed as follows: let t be the coordinate of the interval. Let α be the unique 1-form of pointwise norm 1 (and positively oriented) whose kernel is D. The closed 2-form Ω is defined to be $\omega+d(t\alpha)$, where α and ω represent the pullback of the corresponding forms to $M\times[-\epsilon,\epsilon]$. If ϵ is chosen small enough then Ω is symplectic.

In the points of M the almost complex structure is extended by sending the positively oriented g-unitary vector in $D^\perp$ to $\partial/\partial t$; in those points $\partial/\partial t$ is also defined to have norm 1 and to be orthogonal to TM. It is routine to further extend J to a c.a.c.s. on the symplectization. The metric defined by Ω and the almost complex structure also extends g. We will not use different notation for the extension of the almost complex structure and metric if there is no risk of confusion.

We also fix G a J-complex distribution on the symplectization restricting to D in the points of M. To do that we choose any line field that in the points of M contains $\partial/\partial t$; this line field spans a complex line field. Its orthogonal w.r.t. g is by construction J-complex and extends D. □

Let (M,D,w) be a 2-calibrated submanifold of (P,Ω). Let us fix J a c.a.c.s. on (P,Ω) so that D is J-invariant and $g:=\Omega(\cdot,J)$. The restriction of (J,g) to (M,D) induces an almost CR structure. We also assume the existence of G a J complex distribution that coincides with D at the points of M. The main example to have in mind is the symplectization of (M,D,ω) with an almost complex structure as defined in lemma 4.

We have at our disposal the approximately holomorphic theory for symplectic manifolds [4]. At this point we pause to warn the reader that throughout this subsection and the rest of the paper we will be using A.H. sequences of sections defined in both symplectic (see definitions in [4]) and 2-calibrated manifolds (definition 7). Whenever there is no risk of confusion about the base space we will just speak about A.H. sequences of sections.

Let $(L_\Omega^{\otimes k},\nabla_k)$ be the sequence of powers of the pre-quantum line bundle $(L_\Omega,\nabla)\to(P,\Omega)$. This is a very ample sequence of bundles (in the sense of [4]), and it restricts to a very ample sequence of line bundles $(L^{\otimes k},\nabla_k)\to(M,D,J,g_k)$ (definition 6).

One expects that if $\tau_k\in\Gamma(L_\Omega^{\otimes k})$ is a (symplectic) A.H. sequence of sections, then $\tau_{k|M}\colon M\to L^{\otimes k}$ is also an A.H. sequence of sections (definition 7). Even more, it is possible to construct reference sections by restricting (symplectic) reference sections centered at points of M. The key point to prove these results is the choice of appropriate charts.

Recall that in $\mathbb{C}^p=\mathbb{R}^{2p}$, we denote the foliation whose leaves are associated to g distinguished complex coordinates (resp. d distinguished real coordinates) by $\mathbb{C}^g$ (resp. $\mathbb{R}^d$); their orthogonals are denoted by $\mathbb{C}^{p-g}$ and $\mathbb{R}^{2p-d}$ respectively. From now on if we compare the distance of $\mathbb{C}^g$ to any distribution of the same dimension, we will assume the latter to be transversal to $\mathbb{C}^{p-g}$.

Definition 12. *Let (P,Ω) be a compact symplectic manifold and G a J-complex distribution. A family of (symplectic) approximately holomorphic coordinates (resp. Darboux charts) $\varphi_{k,x}\colon(\mathbb{C}^p,0)\to(U_{k,x},x)$ is said to be adapted to G if*

$$|\mathbb{C}^g - G|_{g_0} \le |(z_k,s_k)|O(k^{-1/2}), \qquad |\mathrm{d}^j(\mathbb{C}^g - G)|_{g_0} \le O(k^{-1/2}),\ \forall j\ge 1$$
$$|\mathbb{C}^{p-g} - G^\perp|_{g_0} \le |(z_k,s_k)|O(k^{-1/2}), \qquad |\mathrm{d}^j(\mathbb{C}^{p-g} - G^\perp)|_{g_0} \le O(k^{-1/2}),\ \forall j\ge 1$$

The existence of approximately holomorphic (resp. Darboux) charts adapted to G is straightforward. Once we have approximately holomorphic (resp. Darboux) charts, we compose with a unitary transformation sending G to $\mathbb{C}^g$ at the origin.

Given a 2-calibrated submanifold $(M,D)\hookrightarrow(P,\Omega)$, in order to select coordinates charts *adapted to* M we fix a distribution $T^{||}M$ defined in a tubular neighborhood of M as follows: the neighborhood is defined by flowing a little bit the geodesics normal to M. For each point y in the neighborhood, let $x\in M$ be the starting point of the unique geodesic normal to M through y. Then $T^{||}_yM$ is the result of parallel translating T_xM along the geodesic.

Definition 13. *Let $(M,D)\hookrightarrow(P,\Omega)$ be a 2-calibrated submanifold and let G and $T^{||}M$ distributions constructed as above. A family of (symplectic) A.H coordinates $\varphi_{k,x}\colon(\mathbb{C}^p,0)\to(U_{k,x},x)$ (centered at every point of P) is adapted to M if they are also adapted to G and for the charts centered at points of M the following conditions hold:*

1. *M sits in each chart as a fixed linear subspace $\mathbb{R}^{2n+1}\times\{0\}\subset\mathbb{C}^p$ and at the origin $D=\mathbb{R}^{2n}\times\{0\}\subset\mathbb{R}^{2n+1}\times\{0\}$, $D^\perp=\{0\}\times\mathbb{R}\subset\mathbb{R}^{2n+1}\times\{0\}$*
2. *$|\mathbb{R}^{2n+1}-T^{||}M|_{g_0}\le|(z_k,s_k)|O(k^{-1/2})$, $|\mathrm{d}^j(\mathbb{R}^{2n+1}-T^{||}M)|_{g_0}\le O(k^{-1/2})$, $\forall j\ge 1$*

We speak of A.H. charts adapted to M and Darboux along M if $\varphi^*_{k,x}\omega_{|M}=\omega_0$.

Lemma 5. *Let $(M,D)\hookrightarrow(P,\Omega)$ be a 2-calibrated submanifold. Then approximately holomorphic charts adapted to M and Darboux along M can always be constructed.*

Proof. We start by fixing approximately holomorphic coordinates adapted to G. Then we forget about the ones centered at points of M, that are going to be substituted by new ones.

For every $x\in M$, we fix initial charts φ_x depending smoothly on the center, at least in a small neighborhood around each point, with $(J,g)=(J_0,g_0)$ at the origin. Then we compose with maps $\Theta_x\colon(\mathbb{C}^p,0)\to(\mathbb{C}^p,0)$ that are tangent to the identity map at the origin and send M to a vector space in $\mathbb{C}^p$. The distribution D at the origin is J_0-complex. By composing with a unitary transformation (D_x,T_xM) can be assumed to be $(\mathbb{C}^n\times\{0\},\mathbb{R}^{2n+1}\times\{0\})\subset\mathbb{R}^{2p}$.

Next we essentially apply lemma 1 on the leaf $\mathbb{R}^{2n+1}\times\{0\}\subset\mathbb{R}^{2p}$ to get Darboux charts for M: let $\Theta_x\colon\mathbb{R}^{2n+1}\to\mathbb{R}^{2n+1}$ be the map which is the identity on $\mathbb{C}^n\times\{0\}$, preserves the foliation by complex hyperplanes and sends the kernel of ω to the "vertical" or real line field in $\mathbb{R}^{2n+1}\times\{0\}$. We extend it to a diffeomorphism of $\mathbb{R}^{2p}$ independently of the coordinates $x^{2n+2},\dots,x^{2p}$. Since the map is by construction tangent to the identity at the origin, we keep the properties at the origin described in the previous paragraph.

We now apply Darboux' lemma on $\mathbb{R}^{2n}\times\{0\}$ for each x. The result is a diffeomorphism on $\mathbb{R}^{2n}$ that can be assumed to preserve J_0 at the origin. We extend it independently of $x^{2n+1},\dots,x^{2p}$ to a diffeomorphism of $\mathbb{C}^p$.

Notice that at the origin $(D,TM)=(\mathbb{R}^{2n}\times\{0\},\mathbb{R}^{2n+1}\times\{0\})$, $J=J_0$, $G\oplus G^\perp=\mathbb{C}^n\oplus\mathbb{C}^{p-n}$ and $\ker\omega_{|D}$ is the Euclidean orthogonal of $\mathbb{R}^{2n}\times\{0\}\subset\mathbb{R}^{2n+1}\times\{0\}$. Hence if we apply the dilatation $\rho_{k^{-1/2}}\colon\mathbb{R}^{2p}\to\mathbb{R}^{2p}$ we obtain a family of charts with the desired properties.

□

Lemma 6. *A family of A.H. charts $\varphi_{k,x}\colon (\mathbb{C}^p, 0) \to (U_{k,x}, x)$ adapted to M and Darboux along M constructed as in lemma 5 restricts to M to Darboux charts.*

Proof. It follows because the charts in lemma 5 are obtained by applying a construction depending smoothly on the center of the chart to obtain a number of equalities for tensors and distributions at the origin, and then rescaling. Hence, when we restrict the charts to M point 1 in definition 8 holds. Points 2 and 3 follow because before rescaling we get the equalities at the origin $D \oplus D^{\perp} = \mathbb{R}^{2n} \oplus \mathbb{R}$ and $J = J_0$. The Darboux condition holds by construction.

□

Lemma 7. *Over $B(0, O(k^{1/2}))$ in the domain of charts as in lemma 5, it is possible to fix a family of unitary trivializations of $\varphi_{k,x}^* L_{\Omega}^{\otimes k}$ with connection forms $A_{k,x}$ such that for all $k \gg 1$*

1. $|A_{k,x}|_{g_0} \le O(|z_k|)$, $|\mathrm{d}A_{k,x}|_{g_0} \le O(1)$, $|\mathrm{d}^j A_{k,x}|_{g_0} \le O(k^{-1/2})$, $j \ge 2$
2. $A_{k,x|M} = A$

Proof. By construction $|\varphi_{k,x}^* k\omega|_{g_0} \le O(1)$, $|\mathrm{d}^j \varphi_{k,x}^* k\omega|_{g_0} \le O(k^{-1/2})$, $j \ge 1$ on $B(0, O(k^{1/2}))$. Hence, we deduce the existence unitary trivializations with connection forms $A'_{k,x}$ satisfying the bounds of condition 1.

When we restrict the connection forms to M they coincide with A up to a exact 1-form $dF_{k,x}$ defined on $\mathbb{R}^{2n+1} \times \{0\}$; its bounds are as in point 1 above, but on $\mathbb{R}^{2n+1} \times \{0\}$ instead of on $\mathbb{C}^p$. We extend it to $\mathbb{C}^p$ independently of the remaining coordinates and still denote it by $F_{k,x}$. It is always possible to find a unitary trivialization $\xi_{k,x}$ of $\varphi_{k,x}^* L_{\Omega}^{\otimes k}$ whose connection form is $A'_{k,x} + dF_{k,x}$. These trivializations give the desired result.

For simplicity we will denote the family by A when there is no risk of confusion.

□

Let G be the J-complex distribution on P that extends D. Given $\tau_k \in \Gamma(L_{\Omega}^{\otimes k})$, the restriction of the covariant derivative of τ_k to G will be denoted by $\nabla_G \tau_k \in \Gamma(G^* \otimes L_{\Omega}^{\otimes k})$. Since G is J-complex, we can write

$$\nabla_G \tau_k = \bar{\partial}_G \tau_k + \partial_G \tau_k, \ \bar{\partial}_G \tau_k \in \Gamma(G^{*0,1} \otimes L_{\Omega}^{\otimes k}), \ \partial_G \tau_k \in \Gamma(G^{*1,0} \otimes L_{\Omega}^{\otimes k})$$

Lemma 8.

1. *If $\tau_k\colon P \to L_{\Omega}^{\otimes k}$ is an A.H. sequence then $\tau_{k|M}\colon M \to L^{\otimes k}$ is also an A.H. sequence.*
2. *Moreover, the restriction of a family of reference sections of $(L_{\Omega}^{\otimes k}, \nabla_k) \to (P, \Omega)$ centered at the points of M (as defined in* [4]*) is a family of reference sections of $(L^{\otimes k}, \nabla_k) \to (M, D, \omega)$.*
3. *If $\tau_k\colon P \to L_{\Omega}^{\otimes k}$ is an A.H. sequence then $\bar{\partial}_G \tau_k \approxeq 0$.*

Proof. We fix a family of A.H. charts adapted to M and Darboux along M and trivialize the bundles $L_{\Omega}^{\otimes k}$ as in lemma 7. Let $x_k^1, \dots, x_k^{2p}$ be the coordinates and write $\tau_{k,x} = f_{k,x} \xi_{k,x}$.

We first observe that lemmas 2 and 3 for symplectic manifolds also hold for the connection forms $A_{k,x}$ provided by lemma 7.

By lemma 6 the restriction of the coordinates to M are Darboux charts. We can apply lemma 2 for almost complex manifolds, bounds of order O(1) and the connection forms provided by lemma 7 to conclude that the partial derivatives of $f_{k,x}$ are bounded by O(1) in the ball $B(0, O(1)) \subset \mathbb{R}^{2p}$. In particular, we get the same bounds if we only take into account the partial derivatives w.r.t. the variables $x_k^1, \dots, x_k^{2n+1}$ and restrict our attention to $B(0, O(1)) \subset \mathbb{R}^{2n+1}$. Now

if we apply back lemma 2 (this time for almost CR manifolds) we conclude that $|\nabla^j(\tau_{k|M})|_{g_0} \leq O(1)$, $\forall j \geq 0$, in $B(0, O(1)) \subset \mathbb{R}^{2n+1}$ and for all $x \in M$, the constants being independent of x. Therefore $|\nabla^j(\tau_{k|M})|_{g_k} \leq O(1)$, $\forall j \geq 0$, in all the points of M.

Lemma 3 for symplectic manifolds and the connections of lemma 7 gives

$$\left| \frac{\partial}{\partial x_k^\alpha}(\bar{\partial}_0 + A_{k,x}^{0,1}) f_{k,x} \right|_{g_0} \leq O(k^{-1/2}), \tag{14}$$

in $B(0, O(1)) \subset \mathbb{R}^{2p}$. Let us consider the splitting $\mathbb{C}^n \times \mathbb{C}^{p-n}$. The operator $\bar{\partial}_0 + A_{k,x}^{0,1}$ and its derivatives can be split into two pieces using it. We consider the part involving $d\bar{z}_k^1, \dots, d\bar{z}_k^n$, for which the above inequalities also hold, but now in $B(0, O(1)) \subset \mathbb{R}^{2n+1}$. Since the restriction of $A_{k,x}$ to $\mathbb{C}^n \times \mathbb{R}$ is A, the restriction to M of the piece of $\bar{\partial}_0 + A_{k,x}^{0,1}$ involving $d\bar{z}_k^1, \dots, d\bar{z}_k^n$ is the operator $\bar{\partial}_0 + A^{0,1}$ of lemma 3. Thus we can apply this lemma (we already have the required bounds for the partial derivatives of $f_{k,x}$) to conclude $\bar{\partial}(\tau_{k|M}) \approxeq 0$, and this proves point 1.

It is also easy to check that reference sections for $L_\Omega^{\otimes k}$ centered at the points of M restrict to reference sections for $L^{\otimes k}$, and hence point 2 also holds.

To proof $\bar{\partial}_G \tau_k \approxeq 0$ we use the previous ideas: equation 14 and lemma 3 give

$$|\nabla^j \bar{\partial}_{\mathbb{C}^n} \tau_k|_{g_0} \leq O(k^{-1/2}),\ \forall j \geq 0,$$

in $B(0, O(1)) \subset \mathbb{R}^{2p}$, where $\bar{\partial}_{\mathbb{C}^n}$ is the part of $\bar{\partial}_0 + A_{k,x}$ involving $d\bar{z}_k^1, \dots, d\bar{z}_k^n$. The choice of A.H. charts adapted to G and the bounds $|\nabla^j \tau_k|_{g_0} \leq O(1)$, $\forall j \geq 0$, easily imply

$$|\nabla^j(\bar{\partial}_{\mathbb{C}^n}\tau_k - \bar{\partial}_G \tau_k)|_{g_0} \leq O(k^{-1/2}),\ \forall j \geq 0$$

and therefore $\bar{\partial}_G \tau_k \approxeq 0$. □

3.2. **Higher rank ample bundles.** So far we have only considered approximately holomorphic theory for the sequence of line bundles $(L^{\otimes k}, \nabla_k) \to (M, D, \omega)$, but there are obvious extensions for sequences of the form $E \otimes L^{\otimes k}$, where E is any hermitian bundle of rank m with compatible connection. Regarding the local theory the role of the reference sections is played by reference basis $\tau_{k,x,1}^{\mathrm{ref}}, \dots, \tau_{k,x,m}^{\mathrm{ref}}$, where each $\tau_{k,x,j}^{\mathrm{ref}}$ is an A.H. sequence with gaussian decay w.r.t. x and they are a frame of E comparable to a unitary one in $B_{g_k}(x, O(1))$. Reference frames are constructed by tensoring reference sections for $L^{\otimes k}$ with local unitary frames of E.

4. Estimated transversality and approximate holomorphic stratifications

Let τ_k be an A.H. sequence of sections of $L^{\otimes k} \to (M, D, \omega)$. Proposition 1 for codimension 2 submanifolds is proved by pulling back the $\mathbf{0}$ section of $L^{\otimes k}$. To obtain W_k a 2-calibrated submanifold τ_k has to be transversal along D, so that $TW_k \cap D$ defines a codimension 1 distribution on W_k. Next, to make sure that $W_k \cap D$ is a symplectic distribution the ratio $|\bar{\partial}\tau_k(x)|/|\partial\tau_k(x)|$ has to be smaller than 1; since $\nabla_D = \bar{\partial} + \partial$, $\nabla_D \tau_k(x)$ has to be asked to be not only to be surjective but to have norm greater than $O(k^{-1/2})$ (estimated transversality).

For each point x we can use the reference sections to turn the local estimated transversality problem over $B_{g_k}(x, \rho)$ into a estimated transversality problem for a sequence of functions $f_{k,x} \colon B(0, \rho) \subset \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}$, $\tau_k = f_{k,x}\tau_{k,x}^{\mathrm{ref}}$ (more generally $\mathbb{C}^m$-valued functions for bundles of rank m). Thus, we have an equivalent local estimated transversality problem for a 1 real parameter family of A.H. functions from $\mathbb{C}^n$ to $\mathbb{C}$. This problem is known to have solution [5, 20].

A local solution furnished by the use of reference sections will hold over the ball $B_{g_k}(x, O(1))$, but the reference section is supported in $B_{g_k}(x, O(k^{1/6}))$. The consequence is that there will be interference among different solutions, but unlike transversality, estimated transversality does behave well under addition.

Let $l\colon \mathbb{R}^a \to \mathbb{R}^b$ be a surjective linear map. Recall that $|l| \geq \eta > 0$ if there exist a right inverse with norm smaller or equal than η^{-1}.

Definition 14. *Let (P, g) be a riemannian manifold, (E, ∇) a hermitian bundle over it and Q_x a subspace of T_xP. We say that $\tau\colon P \to E$ is η-transversal to $\mathbf{0}$ at x along Q_x if either $|\tau(x)| \geq \eta$ or $\nabla_{Q_x}\tau(x)$ is surjective and $|\nabla_{Q_x}\tau(x)| \geq \eta$.*

If Q is a distribution we say that τ is η-transversal to $\mathbf{0}$ along Q if the above condition holds at all the points where Q is defined (for example when Q is the tangent bundle of a submanifold).

Let (M, D, ω) be a 2-calibrated manifold, $E_k := E \otimes L^{\otimes k}$ and $\tau_k\colon (M, g_k) \to (E_k, \nabla_k)$ a sequence of sections. We say that the sequence τ_k is uniformly transversal to $\mathbf{0}$ along D if $k_0 \in \mathbb{N}$, $\eta > 0$ exist such that τ_k is η-transversal to $\mathbf{0}$ along D for $k \geq k_0$.

For a symplectic manifold the definition of uniform transversality along a distribution Q (possibly the tangent bundle to a 2-calibrated submanifold) is analogous.

It is possible to attain estimated transversality using both the intrinsic and the relative point of view. Using the former, what we do is (locally) solving transversality problems for 1-parameter families of A.H. functions from $\mathbb{C}^n$ to $\mathbb{C}^m$.

Regarding the latter we follow the ideas of J.-P Mohsen developed for contact manifolds (see [27], second lemma in subsection 6.1): if in the symplectization $(M \times [-\epsilon, \epsilon], \Omega)$ we are able to find an A.H. sequence τ_k η-transversal along M to $\mathbf{0}$, then for any constant C, $0 < C < \sqrt{2}/2$, there exists $k_0(C)$ such that for any $k \geq k_0$ the section $\tau_{k|M}$ is $C\eta$-transversal to $\mathbf{0}$ along D.

4.1. **Geometric reformulation of estimated transversality.** We recall that in this section we deal with estimated transversality along D in a 2-calibrated manifold (intrinsic theory) or with estimated transversality along a 2-calibrated submanifold M inside a symplectic manifold P (relative theory). Sometimes we might refer to both situations as transversality along a distribution Q in the riemannian manifold P.

Since estimated transversality is achieved by combining local solutions, one expects to be able attain it to sequences of strata S_k that locally look like the zero section of a trivial bundle: the local functions $f_k\colon U_k \subset E_k \to \mathbb{C}^l$, $S_k \cap U_k = f_k^{-1}(\mathbf{0})$, should be approximately holomorphic w.r.t to the almost CR structure in the total space of the bundles $(E_k, \nabla_k) \to (M, D, J, g_k)$ induced by the one on M, the connection and the hermitian metric on E_k, so that $f_k \circ \tau_k$ are A.H. functions (or a weaker property that ensures this last condition). One also has to make sure that out of some estimated transversality for $f_k \circ \tau_k$, it is possible to obtain enough transversality for τ_k to make the standard globalization procedure work.

In the relative context $\tau_k\colon P \to E_k$ the estimated transversality problem along $M \subset P$ (in principle to the $\mathbf{0}$ section) has the same difficulty as the usual estimated transversality problem (this is the work of J.-P. Mohsen [27], section 5). Thus, one expects this principle to be valid in the case of relative estimated transversality to more complicated strata S_k.

To give a global definition of what transversality to a submanifold $S \subset E$ is, we need to recall a more geometric definition of estimated transversality along a distribution Q, together with the following concepts (see [21] for a more detailed exposition).

Definition 15. *Let W be a vector space with non-degenerate inner product so that for any $u, v \in W$ we can compute the angle $\angle(u,v)$. Given $U \in Gr(p,W)$ and $V \in Gr(q,W)$ $p,q>0$, the maximal angle of U and V, $\angle_{\mathrm{M}}(U,V)$, is defined as follows:*

$$\angle_{\mathrm{M}}(U,V) \quad := \quad \max_{u\in U\setminus\{0\}}\min_{v\in V\setminus\{0\}}\angle(u,v)$$

In general the maximal angle is not symmetric, but when $p=q$ it has symmetry and defines a distance in the corresponding grassmannian (see [28]).

The minimum angle between transversal complementary subspaces is defined as the minimum angle between two non-zero vectors, one on each subspace. An extension of this notion for transversal subspaces with non-trivial intersection is:

Definition 16. *(Definition 3.3. in* [28]*) Using the notation of definition 15, $\angle_{\mathrm{m}}(U,V)$ –the minimum angle between non-void subspaces U and V– is defined as follows:*

- *If $\dim U + \dim V < \dim W$, then $\angle_{\mathrm{m}}(U,V) := 0$.*
- *If the intersection is non-transversal, then $\angle_{\mathrm{m}}(U,V) := 0$.*
- *If the intersection is transversal, we consider the orthogonal to the intersection and its intersections U_c and V_c with U and V respectively. We define $\angle_{\mathrm{m}}(U,V) := \min_{u\in U_c\setminus\{0\}}\min_{v\in V_c\setminus\{0\}}\angle(u,v)$.*

The minimum angle is symmetric.

The most important property relating maximal and minimal angle is:

Proposition 3. *(Proposition 3.5 in* [28]*) For non-void subspaces U,V,W of $\mathbb{R}^n$ the following inequality holds:*

$$\angle_{\mathrm{m}}(U,V) \leq \angle_{\mathrm{M}}(U,W) + \angle_{\mathrm{m}}(W,V)$$

Let $\tau\colon P \to E$ be a section of a hermitian bundle with connection and Q a distribution on P. Let us denote the pullback of Q to E by $\hat{Q}$. Let $\mathcal{H}$ be the horizontal distribution associated to the linear connection and let $\mathcal{H}_Q$ denote its intersection with $\hat{Q}$. Finally let $T_Q\tau$ denote the intersection of the tangent bundle to the graph of τ with $\hat{Q}$.

Lemma 9. *When the point $\tau(x)$ is close enough to the* **0** *section, so that estimated transversality is a condition on $\nabla_Q\tau(x)$, then there exists a constant $C>0$ determined by an upper bounds on $|\nabla_Q\tau(x)|, |\tau(x)|$ such that:*

1. $|\nabla_Q\tau(x)| \geq \eta \Rightarrow \angle_{\mathrm{m}}(\mathcal{H}_Q, T_Q\tau) \geq C^{-1}\eta$ *(the angle measured in $\hat{Q}_{\tau(x)}$).*
2. $\angle_{\mathrm{m}}(\mathcal{H}_Q, T_Q\tau) \geq \eta \Rightarrow |\nabla_Q\tau(x)| \geq C\sin\eta$

Proof. Let us assume $Q = TP$. The vector space $T_{\tau(x)}E = \mathcal{H}_{\tau(x)} \oplus T^vE_x$ is endowed with the direct sum metric. We compose with an isometry preserving the direct sum structure so that $\mathcal{H}_{\tau(x)} \oplus T^vE_x$ becomes $\mathbb{R}^a \oplus \mathbb{R}^b$ with the Euclidean metric.

Let $h\colon T\tau(x) \to \mathbb{R}^b$ be the orthogonal projection, which is onto. By lemma 3.8 in [28] (in their notation $T\tau(x)$ becomes U and $\mathbb{R}^a \times \{0\} = \mathcal{H}_{\tau(x)}$ becomes V) if h has a right inverse θ with $|\theta| \leq \eta^{-1}$, then $\angle_{\mathrm{m}}(\mathcal{H}_{\tau(x)}, T\tau(x)) \geq \eta$.

By definition we understand $\nabla\tau(x)$ as a map with target space the fiber $T^vE_x = E_x$ (we project from the tangent space of the total space using the induced metric). Therefore $\nabla\tau = h \circ d\tau(x)$, with $d\tau(x)\colon T_xP \to T\tau(x)$ the usual derivative, which is an isomorphism.

Now if θ' is a right inverse for $\nabla\tau(x)$, $|\theta'| \leq \eta^{-1}$, then $d\tau(x) \circ \theta'$ is a right inverse for h with norm bounded by $|d\tau(x)|\eta^{-1}$. Thus, by lemma 3.8 in [28] $\angle_{\mathrm{m}}(\mathcal{H}_{\tau(x)}, T\tau(x)) \geq |d\tau(x)|^{-1}\eta$.

Conversely, the projection h has always an right inverse θ of minimum norm. Let $W := T\tau(x) \cap \mathcal{H}_{\tau(x)}$ and $U_c := T\tau(x) \cap W^{\perp}$. If we compose θ with the orthogonal

projection $T\tau(x) \to U_c$, we obtain a right inverse $\hat{\theta}$ for $h_{|U_c}$ such that $|\theta| = |\hat{\theta}|$. If now $\angle_{\mathrm{m}}(\mathcal{H}_{\tau(x)}, T\tau(x)) \geq \eta$, then the equation involving inequalities of lemma 3.8 in [28] implies $|\hat{\theta}| \leq (\sin\eta)^{-1}$. Hence $d\tau(x)^{-1} \circ \theta$ is a right inverse for $\nabla\tau(x)$ with norm bounded by $|d\tau(x)|^{-1}(\sin\eta)^{-1}$.

When Q is not TP, we fix an isometry sending $(\mathcal{H}_Q, \mathcal{H})$ at $\tau(x)$ to $(\mathbb{R}^{a'} \times \{0\}, \mathbb{R}^a)$ with the Euclidean metric, and apply the above arguments to $\mathbb{R}^{a'} \oplus \mathbb{R}^b$.

We have $C = |d_Q\tau(x)|$, with $d_Q\tau(x)$ the restriction of $d\tau(x)$ to Q_x. Notice that a bound for $|d_Q\tau(x)|$ can be obtained from bounds for $|\tau(x)|$ and $|\nabla_Q\tau(x)|$. □

Remark 5. *In the definition of minimum angle $\angle_{\mathrm{m}}(U, V)$, when U, V are not complementary we work with the intersections in $(U \cap V)^{\perp}$ where we can apply the usual notion of minimum angle for complementary subspaces. Instead of $(U \cap V)^{\perp}$ one might choose any other subspace W complementary to $U \cap V$ to give a different notion of minimum angle. In certain situations this is a good strategy because there are natural complementary subspaces available. It is easy to see that the new notion of minimum angle is comparable to the one of definition 16, the comparison given by multiplying by a constant depending only on $\angle_{\mathrm{m}}(U \cap V, W)$ (there is no ambiguity since these are complementary subspaces). Actually, those new notions depending on the complementary coincide with the one given in 16, but for a new metric, which is comparable to the Euclidean in terms of $\angle_{\mathrm{m}}(U \cap V, W)$ (very much as it happened with the isomorphim $d_Q\tau(x)$ in the previous lemma).*

We need a second result relating angles and intersections.

Lemma 10. *Let U, V, W linear subspaces of $\mathbb{R}^n$ such that $\angle_{\mathrm{m}}(V, W) \geq \gamma > 0$. Let $\angle_{\mathrm{M}}(U, V) \leq \delta$. Then there exists $C(\gamma, \mathrm{dim}V, n) > 0$ such that*

$$\angle_{\mathrm{M}}(U \cap W, V \cap W) \leq C\delta$$

Proof. For each $u \in U \backslash \{0\}$, we have $\angle(u, V) = \angle(u, h(u))$, where $h\colon \mathbb{R}^n \to V$ is the orthogonal projection. We consider a new complementary subspace to V: let V_W be the orthogonal to $V \cap W$ in W, and define $h_W : \mathbb{R}^n \to V$ to be the projection along V_W. It follows that $\angle(u, h_W(u)) \leq C\angle(u, h(u)) = C\angle(u, V)$, and by construction if $u \in U \cap W$, then $\angle(u, h_W(u)) = \angle(u, V \cap W)$.

□

Let $S \subset E$ be a submanifold in the total space of the vector bundle E over either a 2-calibrated or a symplectic manifold, transversal to the fibers. Let $\hat{g}$ be the metric in E induced by the connection, the bundle metric and the metric g in the base. The submanifold may be open and have a bad behavior near its boundary $\partial S = \bar{S} \backslash S$, so given $\bar{\eta}$ we consider the points in S at distance greater than $\bar{\eta}$ from the boundary, the points of S *$\bar{\eta}$-far from the boundary.* For any $\eta > 0$ and typically much smaller than $\bar{\eta}$, let $\mathcal{N}_S(\eta, \bar{\eta})$ be tubular neighborhood of radius η of the points $\bar{\eta}$-far from the boundary, and define $T^{||}S$ in $\mathcal{N}_S(\eta, \bar{\eta})$ as follows: parallel translate TS along the geodesics orthogonal to S starting in the points $\bar{\eta}$-far from the boundary of S.

$T^{||}S$ plays the role of $\mathcal{H}$. We use the notation $T^{||}_Q S := T^{||}S \cap \hat{Q}$.

Definition 17. *τ is $(\eta, \bar{\eta})$-transversal to S along Q at x if either $\tau(x)$ misses the union of S with $\mathcal{N}_S(\eta, \bar{\eta})$, or $\tau(x)$ enters in $\mathcal{N}_S(\eta, \bar{\eta})$ so that $\angle_{\mathrm{m}}(T_Q\tau, T^{||}_Q S) \geq \eta$ at $\tau(x)$, or $\tau(x)$ intersects S in the points $\bar{\eta}$-close to the boundary with $\angle_{\mathrm{m}}(T_Q\tau, T_Q S) \geq \bar{\eta}$.*

Uniform transversality of τ_k along Q to S_k is defined as $(\eta, \bar{\eta})$-transversality for some $\eta, \bar{\eta} > 0$ and for all $k \gg 1$.

Conditions on a sequence of submanifolds S_k of complex codimension l (or more generally on stratifications) can be imposed, so that local estimated transversality along Q of $\tau_{k,x}$ over $B_{g_k}(x, O(1))$ to the points of S_k far from ∂S_k is equivalent to estimated transversality along Q of a related $\mathbb{C}^l$-valued function to $\mathbf{0}$ (lemma 11).

We will consider stratifications $\mathcal{S} = (S_k^a), a \in A_k$. The stratification will be required to be finite in the sense that $\#(A_k)$ must be bounded independently of k and the boundary of each strata $\partial S_k^b = \bar{S}_k^b \backslash S_k^b$ will be the union of the strata of smaller dimension

$$\partial S_k^b = \bigcup_{a<b} S_k^b$$

Definition 18. *Let $E_k = E \otimes L^{\otimes k} \to (M, D, J, g_k)$ and $(S_k^a)_{a\in A_k}$ finite stratifications of E_k whose strata are transversal to the fibers. Let $r \in \mathbb{N}$, $r \geq 2$. The sequence of strata is Whitney C^r-approximately holomorphic (C^r-A.H.) if for any bounded open set U_k of the total space of E_k and any $\epsilon > 0$, constants $C_\epsilon, \rho_\epsilon > 0$ only depending on ϵ and on the size of U_k –but not on k– can be found, so that for any point $y \in U_k$ in a strata S_k^a for which $d_{\hat{g}_k}(y, \partial S_k^a) > \epsilon$, there exist complex valued functions $f_1, \ldots, f_l$ such that $B_{\hat{g}_k}(y, \rho_\epsilon) \cap S_k^a$ is given $f_1 = \cdots = f_l = 0$, and the following properties hold:*

1. *(Uniform transversality to the fibers + transversal comparison) The restriction of $df_1 \wedge \cdots \wedge df_l$ to $T^v E_k$ is bounded from below by ρ_ϵ.*
2. *(Approximate holomorphicity along the fibers) The restriction of the function $f = (f_1, \ldots, f_l)$ to each fiber is C^r-A.H.(C_ϵ, k).*
3. *(Horizontal approximate holomorphicity + holomorphic variation of the restriction to the fiber + estimated variation of the restriction to the fiber) For any λ, k, and τ C^r-A.H.(λ, k) local section of E_k with image cutting $B_{\hat{g}_k}(y, \rho_\epsilon)$, $f_j \circ \tau$ is C^r-A.H.$(\lambda C_\epsilon, k)$. Moreover, if θ is a local C^r-A.H.(λ, k) section of $\tau^* T^v E_k$, $df_\tau(\theta)$ is C^r-A.H.$(\lambda C_\epsilon, k)$.*
4. *(Estimated Whitney condition) For each $\eta > 0$ small enough, there exists $\delta(\eta) > 0$ such that $\forall y \in S_k^b$ at distance smaller than δ of $S_k^a \subset \partial S_k^b$, $\angle_{\mathrm{M}}(T^{||} S_k^a, T S_k^b)$ at y is bounded by η.*

Remark 6. *For the main applications of our theory (actually only if we use the intrinsic theory) we will need stratifications all whose derivatives are controlled (A.H. stratifications).*

Remark 7. *If we give the corresponding definition using as base space an almost complex manifold instead of an almost CR manifold, we almost recover the definition 3.2 in* [4] *(our condition 4 is a bit weaker).*

Condition 1 is equivalent to the strata have minimum angle with the fibers bounded from below. We just try to mimic the picture of the $\mathbf{0}$ section w.r.t. the fibers of a vector bundle, in which case we even have orthogonality.

Conditions 2 and 3 guarantee that if $\tau_k \colon M \to E_k$ is A.H., then the corresponding $\mathbb{C}^l$-valued function to be made transversal to $\mathbf{0}$ is A.H.

Condition 4 is an estimated Whitney (A) condition.

Though so far we have only considered sequences of vector bundles $E_k \to (M, D, J, g_k)$, we can apply the theory to more general sequences of fiber bundles F_k with fiber an almost complex manifold (in principle compact and equipped with a compatible metric) and a connection in the bundle compatible with the metric and almost complex structure on the fiber. However, we will only deal with projectivizations of very ample sequences of vector bundles; by choosing homogeneous coordinates of projective space and the associated affine charts, these bundles will

be the result of patching vector bundles and all the notions we use will be seen to be independent of the affine charts (the corresponding vector bundle).

Lemma 11. *Let S^a_k be a sequence of strata as those in the stratifications of definition 18 for the base space P either an almost CR manifold (intrinsic theory) or an almost complex manifold (relative theory). Let $\epsilon > 0$ and $0 < \eta \ll \epsilon$. Let $y \in E_k$ be a point in the stratum ϵ-far from the boundary, and let $f_1, \ldots, f_l$ the corresponding local functions defining the stratum in $B_{\hat{g}_k}(y, \rho_\epsilon)$. Let τ_k be a section of E_k whose graph enters in $B_{\hat{g}_k}(y, \rho_\epsilon)$. Then there exist constants $\rho'(\epsilon, \eta, |\tau_k|)$, $C(\epsilon, |\nabla_Q \tau_k|, |\tau_k|), C'(\epsilon, |\nabla_Q \tau_k|, |\tau_k|) > 0$ such that:*

1. *If $\angle_{\mathrm{m}}(T_Q\tau, T^{||}_Q S^a) \geq \eta$ in $B_{\hat{g}_k}(y, \rho_\epsilon)$ then $|d_Q(f \circ \tau)| \geq C\sin(\eta/2)$ in $B_{\hat{g}_k}(y, \rho')$.*
2. *If $|d_Q(f \circ \tau)| \geq \eta$ in $B_{\hat{g}_k}(y, \rho_\epsilon)$, then $\angle_{\mathrm{m}}(T_Q\tau, T^{||}_Q S^a) \geq {C'}^{-1}\eta$ in $B_{\hat{g}_k}(y, \rho')$.*

Proof. By simplicity we omit the subindices for the sections τ_k, the bundles and strata.

Let us assume $\angle_{\mathrm{m}}(T_Q\tau, T^{||}_Q S^a) \geq \eta$.

Step1: Show the existence of $\rho'(\epsilon, \eta, |\tau|) > 0$ such that $\angle_{\mathrm{m}}(T_Q\tau, \ker df \cap \hat{Q}) \geq \eta/2$ in $B_{\hat{g}_k}(y, \rho')$.

According to proposition 3 (proposition 3.5 in [28])

$$\angle_{\mathrm{m}}(T^{||}_Q S^a, T_Q\tau) \leq \angle_{\mathrm{M}}(T^{||}_Q S^a, \ker df \cap \hat{Q}) + \angle_{\mathrm{m}}(\ker df \cap \hat{Q}, T_Q\tau),$$

so we need to prove the existence of $\rho' > 0$ so that in $B_{\hat{g}_k}(y, \rho')$

$$\angle_{\mathrm{M}}(T^{||}_Q S^a, \ker df \cap \hat{Q}) \leq \eta/2 \tag{15}$$

Condition (1) in definition 18 implies $\angle_{\mathrm{m}}(\ker df, \hat{Q}) \geq \gamma(\epsilon)$. If we find $\rho' > 0$ such that in $B_{\hat{g}_k}(y, \rho')$

$$\angle_{\mathrm{M}}(T^{||} S^a, \ker df) \leq C(\gamma(\epsilon))^{-1}\eta/2, \tag{16}$$

we can apply lemma 10, where $U = T^{||}S^a$, $V = \ker df$, $W = \hat{D}$, to conclude that equation 15 holds.

Equation 16 is proven using appropriate charts. The situation we are trying to mimic is that of a locally trivialized vector bundle and we measure the maximal angle between the parallel copies of the $\mathbf{0}$ section (here the leaves of $\ker df$) and $\mathcal{H}$ (here $T^{||}S^a$).

Due to the bounds in definition 18 we can find a chart $\Phi_y \colon \mathbb{R}^a \to B_{\hat{g}_k}(y, \rho_\epsilon)$ such that in $B(0, O(1)) \subset \mathbb{R}^a$ (i) the metrics g_0 and $\Phi^*_y \hat{g}_k$ (that we write $\hat{g}_k$ if it is clear that we work in the chart) are comparable, and the Christoffel symbols of $\hat{g}_k$ are bounded by $O(1)$ (the bounds being uniform on y and k), and (ii) the foliation $\ker df$ is sent to the foliation $\mathbb{R}^{a-2l}$. In $B(0, O(1)) \subset \mathbb{R}^a$ the stratum S becomes $\mathbb{R}^{a-2l} \times \{0\}$ and tubular neighborhoods for $\hat{g}_k$ and g_0 are comparable. At any point q in the neighborhood, a vector in $u \in T^{||}S$ is the result of parallel translating (with $\hat{g}_k$) a vector v in $\mathbb{R}^{a-2l} \times \{0\}$ over $y' \in \mathbb{R}^{a-2l} \times \{0\}$ along the corresponding $\hat{g}_k$-geodesic. Since the Christoffel symbols are bounded, $\angle(u, v)$ is bounded by $e^{\Gamma t} - 1$, $\Gamma > 0$. So by decreasing t, the distance of q to S, we bound the maximal angle by $C(\gamma)^{-1}\eta/2$. Therefore, the final radius ρ' depends on η, on ϵ (because $C(\gamma)$ depends on ϵ) and on how g_0 and $\hat{g}_k$ are related (to order one). This final relation depends on f (and hence on ϵ) and on the metric $\hat{g}_k$ (and hence on $|\tau|$).

Step 2: Show that $\angle_{\mathrm{m}}(T_Q\tau, \ker df \cap \hat{Q}) \geq \eta/2 \Rightarrow |d_Q(f \circ \tau)| \geq C(\epsilon, |\nabla_Q \tau|, |\tau|)\sin(\eta/2)$.

According to lemma 3.8 in [28] (or to lemma 9) the orthogonal projection $h \colon T_Q\tau \to (\ker df \cap \hat{Q})^{\perp}$ has a right inverse with norm bounded by $(\sin(\eta/2))^{-1}$.

Let V_E denote the orthogonal in the fiber T^vE of $(\ker df \cap \hat{Q}) \cap T^vE$. Due to condition 1 in definition 18, this is a subspace complementary to $\ker df \cap \hat{Q}$ and with minimal angle bounded from below in terms of ρ_ϵ and hence in terms of ϵ.

Let $h_E \colon T_Q\tau \to V_E$ be the projection along $\ker df \cap \hat{Q}$. It follows that there is a constant $C_1(\epsilon)^{-1} > 0$ and a right inverse for h_E with norm bounded by $C_1(\epsilon)^{-1}(\sin\eta/2)^{-1}$. We now define

$$h'' = df \circ h_E \circ d_Q\tau \colon Q \to \mathbb{C}^l$$

By construction $h'' = d_Q(f \circ \tau)$. Condition 1 about the restriction of df to the fiber implies the existence of a right inverse for h'' with norm bounded by $|d_Q\tau|^{-1}C_2(\epsilon)^{-1}C_1(\epsilon)^{-1}(\sin\eta/2)^{-1}$. Therefore,

$$|d_Q(f \circ \tau)| \geq C(\epsilon, |d_Q\tau|)\sin\eta/2 \;\text{ in }\; B_{\hat{g}_k}(y, \rho'(\epsilon, \eta, |\tau|))$$

If we now have $|d_Q(f \circ \tau)| \geq \eta$ in $B_{\hat{g}_k}(y, \rho_\epsilon)$, step 2 above implies

$$|h_E \circ d_Q\tau| \geq {C_1'}^{-1}(\epsilon)2\eta$$

Point 1 in lemma 9 gives

$$\angle_{\mathrm{m}}(T_Q\tau, \ker df \cap \hat{Q}) \geq C'(\epsilon, |\nabla_Q\tau|, |\tau|)2\eta,$$

and combined with step 1 we conclude

$$\angle_{\mathrm{m}}(T_Q\tau, T_Q^{||}S^a) \geq C'(\epsilon, |d_Q\tau|)\eta \;\text{ in }\; B_{\hat{g}_k}(y, \rho(\epsilon, \eta, |\tau|))$$

Observe that the constants C, C' grow very large as ϵ and η tend to zero. □

Remark 8. *Notice that the previous lemma does not involve almost complex structures at all. Hence it also holds for arbitrary hermitian bundles, sections and strata which fulfill condition 1 in definition 18.*

Using again appropriate choices of complementary subspaces to measure angle we obtain the following result.

Lemma 12. *Let $\mathcal{S} = (S_k^a)_{a \in A}$ be a sequence of approximately holomorphic stratifications as in definition 18. Assume that the sequence τ_k is uniformly transversal to $\mathcal{S}$ along the directions of a distribution Q whose dimension is greater of equal than the codimension of the strata, and that the uniform bound $|\nabla\tau_k|_{g_k} \leq O(1)$ holds. Then for each $a \in A$, $\tau_k^{-1}(S_k^a)$ is a subvariety of M uniformly transversal to Q.*

Proof. See [21]. □

In particular, the following corollary is deduced.

Corollary 5. *Let $\mathcal{S} = (S_k^a)_{a \in A}$ be a sequence of A.H. stratifications over the 2-calibrated manifold (M, D, ω) as in definition 18. Assume that the A.H. sequence τ_k is uniformly transversal to $\mathcal{S}$ along D. Then for each $a \in A_k$, $\tau_k^{-1}(S_k^a)$ is either empty –if the codimension of S_k^a is bigger than the dimension of D (or M)– or a subvariety uniformly transversal to D.*

For a symplectic manifold transversality along the directions of a (compact) subvariety Q implies that either $\tau_k^{-1}(S_k^a)$ is at g_k-distance of Q bounded from below or it is a subvariety (at least defined in a g_k-neighborhood of Q) uniformly transversal to Q.

Corollary 5 for 2-calibrated manifolds is equivalent to saying that uniform transversality along D implies uniform full transversality. The converse is also true, extending therefore Mohsen's relative transversality result to appropriate sequences of stratifications.

Definition 19. *Let $\mathcal{S}$ be as in definition 18 (over either a 2-calibrated or a symplectic manifold). Then τ_k is uniformly transversal along Q to $\mathcal{S}$ if there exists strictly positive numbers $(\eta_a, \bar{\eta}_a)$ for all $a \in A_k$ such that:*

1. *For all $a \in A_k$ and for all $k \gg 1$, τ_k is $(\eta_a, \bar{\eta}_a)$-transversal along Q to S_k^a.*
2. *For each b, $\bigcup_{a<b} \mathcal{N}_{S_k^a}(\eta_a, \bar{\eta}_a)$ contains the points of S_k^b $\bar{\eta}_b$-close to ∂S_k^b.*

Corollary 6. *Let $\mathcal{S} = (S_k^a)_{a\in A_k}$ be a sequence of A.H. stratifications over the 2-calibrated manifold (M, D, ω) as in definition 18. Assume that the A.H. sequence τ_k is uniformly transversal to $\mathcal{S}$ along TM, for suitable constants $(\eta_a, \bar{\eta}_a)$, $a \in A_k$. Then τ_k is also uniformly transversal to $\mathcal{S}$ along D.*

Proof. By induction we can assume that τ_k is uniformly transversal along D to S_k^a, for every $a < b$.

Let $q \in S_k^b$, with $\tau_k(x) = q$ $\bar{\eta}'$-close to ∂S_k^b. We want to show

$$\angle_{\mathrm{m}}(T_D\tau_k(x), T_D S_k^b(q),) \geq \bar{\eta}',$$

and we will do it applying for some index $a \in A_k$ the inequality

$$\angle_{\mathrm{m}}(T_D\tau_k(x), T_D^{||} S_k^a(q)) \leq \angle_{\mathrm{M}}(T_D^{||} S_k^a(q), T_D S_k^b(q)) + \angle_{\mathrm{m}}(T_D\tau_k(x), T_D S_k^b(q)) \quad (17)$$

If $\bar{\eta}'$ is small enough condition 2 in definition 19 implies the existence of an index $a \in A_k$ such that $q \in \mathcal{N}_{S_k^a}(\eta_a, \bar{\eta}_a)$. If we apply induction we conclude $\angle_{\mathrm{m}}(T_D\tau_k(x), T_D^{||} S_k^a(q)) \geq \eta_a$, so we only need to make $\angle_{\mathrm{M}}(T_D^{||} S_k^a(q), T_D S_k^b(q)) \ll \eta_a$; this is done using lemma 10 with $U = T^{||}S^a(q)$, $V = TS^b(q)$, $W = \hat{D}$. We need to check

$$\angle_{\mathrm{M}}(T^{||}S_k^a(q), TS_k^b(q)) \quad \ll \quad \eta_a \quad (18)$$

$$\angle_{\mathrm{m}}(TS^b(q), \hat{D}) \quad \geq \quad \gamma \quad (19)$$

Equation 18 follows by the estimated Whitney condition by taking $\bar{\eta}'$ small enough; equation 19 uses again the inequality of proposition 3

$$\angle_{\mathrm{m}}(\hat{D}, T^{||}S_k^a(q)) \leq \angle_{\mathrm{M}}(T^{||}S_k^a(q), TS_k^b(q)) + \angle_{\mathrm{m}}(\hat{D}, TS_k^b(q)),$$

together with $\angle_{\mathrm{m}}(\hat{D}, T^{||}S_k^a(q)) \geq 2\gamma$ (by condition 1 in definition 18) and equation 18.

So far we deduced some $\bar{\eta}'$-transversality only in the points $\bar{\eta}'$-close to the boundary of S_b^k.

Now let us assume that for some $\eta > 0$, $\angle_{\mathrm{m}}(T\tau_k(x), T^{||}S_k^b(x)) \geq \eta$ in the tubular neighborhood $\mathcal{N}_{S_k^b}(\eta, \bar{\eta}')$ (here comes the requirement on the constants controlling the transversality along TM, i.e. in those points $\bar{\eta}'$-far from the boundary we need to make sure that $\angle_{\mathrm{m}}(T\tau_k(x), T^{||}S_k^b(x))$ is uniformly bounded from below).

If $\tau_k(x) \in \mathcal{N}_{S_k^b}(\eta, \bar{\eta}')$ then by lemma 11 η-transversality implies η'-transversality to $\mathbf{0}$ of the function $f \circ \tau_k \colon B_{g_k}(x, O(1)) \to \mathbb{C}^l$. From the approximate holomorphicity of the composition $f \circ \tau_k$, for all $k \gg 1$ we deduce $\frac{\sqrt{2}}{3}\eta'$-transversality along D, which again by lemma 11 gives η''-transversality to S_k^b along D (we suppose $\eta'' \leq \eta$).

Therefore, it follows that τ_k is $(\eta'', \bar{\eta}')$-transversal to S_k^b along D. □

5. Pseudo-holomorphic jets

The main applications of the theory of approximately holomorphic geometry for 2-calibrated manifolds are deduced from the existence of generic rank m linear systems.

Let us assume that $(M, \mathcal{D}, J)$ is a Levi-flat CR manifold, and $L \to M$ a positive CR line bundle. Let $\underline{\mathbb{C}}^m \to M$ denote the trivial (and trivialized) bundle of rank m endowed with the trivial connection.

Definition 20. *A CR section $\tau\colon M \to \underline{\mathbb{C}}^{m+1} \otimes L$ (or a rank m linear system of L) is r-generic if its zero set B is a CR submanifold of the expected dimension, and the projectivization $\phi\colon M\backslash B \to \mathbb{CP}^m$ is a leafwise r-generic holomorphic map, i.e. when restricted to each leaf it is transversal to the Thom-Boardman stratification of the bundle of holomorphic r-jets of maps from the leaf to $\mathbb{CP}^m$.*

The existence of r-generic linear systems (possibly of large enough powers of L) has not yet been proved.

The strong transversality problem for a CR function $\phi\colon M \to \mathbb{CP}^m$ to be r-generic is easy to state: we consider $\mathcal{J}^r_{\mathcal{D}}(M, \mathbb{CP}^m)$ the bundle of *CR r-jets* (of foliated holomorphic r-jets) for CR maps from M to $\mathbb{CP}^m$. This bundle admits a CR Thom-Boardman stratification $\mathbb{P}\Sigma$, which restricts to each leaf to the corresponding holomorphic Thom-Boardman stratification. A CR function ϕ is r-generic if and only if its CR r-jet $j^r_{\mathcal{D}}\phi\colon M \to \mathcal{J}^r_{\mathcal{D}}(M, \mathbb{CP}^m)$ (which by definition is the foliated holomorphic r-jet) is transversal along $\mathcal{D}$ to $\mathbb{P}\Sigma$.

Assume that our CR submanifold embeds holomorphically in some complex manifold P and that $\mathcal{D}$ extends to a holomorphic foliation $\mathcal{G}$. There is a canonical submersion $p_{\mathcal{G}}\colon \mathcal{J}^r(P, \mathbb{CP}^m) \to \mathcal{J}^r_{\mathcal{G}}(P, \mathbb{CP}^m)$ from full holomorphic r-jets to foliated ones. The foliated Thom-Boardman stratification $\mathbb{P}\Sigma \subset \mathcal{J}^r_{\mathcal{G}}(P, \mathbb{CP}^m)$ restricts over M to the CR Thom-Boardman stratification $\mathbb{P}\Sigma$ of $\mathcal{J}^r_{\mathcal{D}}(M, \mathbb{CP}^m)$. Let us denote the pullback ${p_{\mathcal{G}}}^{-1}(\mathbb{P}\Sigma)$ by $\mathbb{P}\Sigma^{\mathcal{G}}$.

It is an elementary fact that $j^r_{\mathcal{G}}\phi \in \Gamma(\mathcal{J}^r_{\mathcal{G}}(P, \mathbb{CP}^m))$ -the holomorphic r-jet along $\mathcal{G}$- is transversal along $\mathcal{G}$ to $\mathbb{P}\Sigma$ in the points of M if and only if $j^r\phi \in \Gamma(\mathcal{J}^r(P, \mathbb{CP}^m))$ is transversal along $\mathcal{G}$ to $\mathbb{P}\Sigma^{\mathcal{G}}$ in M. By the results of the previous section, this is equivalent to being transversal to $\mathbb{P}\Sigma^{\mathcal{G}}$ along M.

To obtain an r-generic linear system there is an additional complication coming from the base locus. We first need to make sure that $\tau\colon P \to \underline{\mathbb{C}}^{m+1} \otimes L$ is transversal along M to the zero section, and then solve the r-genericity problem for the projectivization (in a compact region of $P\backslash\tau^{-1}(\mathbf{0})$). Instead of working first with the section τ and then with the projectivization, following ideas of D. Auroux [4] we restate the whole issue as a unique transversality problem *along M* for the *pseudo-holomorphic r-jet extension of* τ, a section of a vector bundle $\mathcal{J}^r(\underline{\mathbb{C}}^{m+1} \otimes L)$.

5.1. **The integrable case.** Let $E \to P$ be a hermitian bundle with compatible connection ∇ over a *complex manifold*, and whose curvature verifies $F^{0,2}_{\nabla} = 0$. The total space of the bundle is a complex manifold (theorem 2.1.53 in [12]) and there is a notion of holomorphic section and hence of holomorphic r-jet. The space of r-jets has natural charts obtained out of holomorphic coordinates in the base and a holomorphic trivialization of the bundle. They provide a local identification of the holomorphic r-jets with $\mathcal{J}^r_{n,m}$, the usual r-jets for holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$.

Let ∂_0 be the Cauchy-Riemann operator defined (locally) using the canonical structure J_0 in the base (the chart) and the trivial connection d in $\underline{\mathbb{C}}^m$. The connection on the fiber bundle can be used to give a different notion of local holomorphic r-jet (in principle chart dependent) by just considering the operator ∂_{∇}: if the connection matrix in the trivialization is $A_x = A^{1,0}_x$, then the *coupled* 1*-jet* of a holomorphic section τ is defined to be $(\tau, \partial_0\tau + A_x\tau))$. Higher order coupled jets are constructed by induction using the connection induced by the flat metric and ∇.

We notice that locally for the above choice of coordinates and trivialization of the bundle, both the usual r-jets and coupled r-jets fill the bundle $(\sum_{j=0}^{r}(T^{*1,0}\mathbb{C}^n)^{\odot j})\otimes \underline{\mathbb{C}}^m = \mathcal{J}^r_{m,n}$, where $\odot$ stands for the symmetric part of the tensor product and $(T^{*1,0}\mathbb{C}^n)^0 \otimes \underline{\mathbb{C}}^m$ for $\underline{\mathbb{C}}^m$. This is due to the existence through any point of E of holomorphic frames tangent to the horizontal distribution of the connection, together with the vanishing $F^{2,0}_{\nabla}$ (the latter implying that $\mathrm{d}A$ and its derivatives are symmetric tensors when evaluated on $(1,0)$-vectors).

For Levi-flat CR manifolds the local model for the pseudo-holomorphic jets to be introduced is the following: the base space is $(\mathbb{C}^n \times \mathbb{R}, J_0, g_0)$ (or rather a ball of Euclidean radius $O(1)$), the bundle *is assumed* to be trivialized by a CR frame and the curvature is of type $(1,1)$. The bundle of CR r-jets is denoted by $\mathcal{J}^r_{D_h,n,m}$ (foliated holomorphic r-jets along D_h); its fiber over each point is that of $\mathcal{J}^r_{n,m}$. There is an obvious notion of CR coupled r-jet. The hypothesis on the trivialization and on the curvature imply that they are also symmetric, so they fill the bundle $\mathcal{J}^r_{D_h,n,m} = \mathcal{J}^r_{n,m}\times\mathbb{R}$.

Using Darboux charts and suitable trivializations this model will be achieved in an approximate way in the theory for 2-calibrated manifolds.

There is a final local model we wish to introduce that would appear in Kähler manifolds P with a holomorphic foliation $\mathcal{G}$. Locally, we have holomorphic coordinates $\mathbb{C}^g\times\mathbb{C}^{p-g}$ with $\mathcal{G}$ sent to $\mathbb{C}^g$ (the foliation with leaves $\mathbb{C}^g\times\{\cdot\}$), and we work with foliated coupled jets along the leaves of $\mathbb{C}^g$. The corresponding bundle of coupled foliated r-jets is denoted by $\mathcal{J}^r_{\mathbb{C}^g,p,m}$. It coincides with $\mathcal{J}^r_{g,m}\times\mathbb{C}^{p-g}$. Transversality problems for this bundle will be transferred to transversality problems in $\mathcal{J}^r_{p,m}$, so we need no further analysis of its properties, though we will be interested at some point in studying the natural submersion $\mathcal{J}^r_{p,m}\to\mathcal{J}^r_{\mathbb{C}^g,p,m}$. This local model is achieved in an approximate way in a symplectic manifold (with c.a.c.s. and metric) with a J-complex distribution G, by using approximate holomorphic charts adapted to G.

5.2. **Pseudo-holomorphic jets.** Let us denote the sequence of bundles $E\otimes L^{\otimes k}\to (M,D,\omega)$ by E_k. We define the bundles

$$\mathcal{J}^r_D E_k := (\sum_{j=0}^{r}(D^{*1,0})^{\odot j})\otimes E_k$$

The hermitian metric $g_{k|D}$ with the one on E_k induce a hermitian metric on $\mathcal{J}^r_D E_k$. Similarly, the Levi-Civita connection induces a connection on D^* (using the metric to see $D^*\hookrightarrow T^*M$ and then projecting $T^*M\to D^*$) and therefore in $D^{*1,0}$ (using the splitting $D^{*1,0}+D^{*0,1}$); combined with the connection on E_k they define connections $\nabla_{k,r}$. Here we also use the symmetrization map

$$\mathrm{sym}_j\colon (D^{*1,0})^{\otimes j}\to (D^{*1,0})^{\odot j} \tag{20}$$

The definition of pseudo-holomorphic r-jets along D (or just pseudo-holomorphic r-jets) for a sequence E_k of hermitian vector bundles is the following (see [4]):

Let $j^{r-1}_D\tau_k\in\mathcal{J}^{r-1}_D E_k$ be the $r-1$-jet of τ_k. It has homogeneous components of degrees $0,1,\ldots,r-1$. We will denote the homogeneous component of degree $j\in\{0,\ldots,r-1\}$ by $\partial^j_{\mathrm{sym}}\tau_k\in\Gamma((D^{*1,0})^{\odot j}\otimes E_k)$.

The connection $\nabla_{k,r-1}$ is actually a direct sum of connections defined on the direct summands $(D^{*1,0})^{\odot j}\otimes E_k$, $j=0,\ldots,r-1$. For simplicity and if there is no risk of confusion we will use the same notation for the restriction of $\nabla_{k,r-1}$ to each of the summands.

The restriction of $\nabla_{k,r-1}\partial^{r-1}_{\mathrm{sym}}\tau_k$ to D defines a section $\nabla_{k,r-1,D}\partial^{r-1}_{\mathrm{sym}}\tau_k\in\Gamma(D^*\otimes (D^{*1,0})^{\odot r-1}\otimes E_k)$. For each $x\in M$, it is a form on D with values in the complex

vector space $(D^{*1,0})^{\odot r-1} \otimes E_k$. Therefore, we can consider its $(1,0)$-component $\partial\partial^{r-1}_{\mathrm{sym}}\tau_k \in \Gamma(D^{*1,0} \otimes (D^{*1,0})^{\odot r-1} \otimes E_k)$. By applying the symmetrization map sym_r of equation 20 we obtain $\partial^r_{\mathrm{sym}}\tau_k \in \Gamma((D^{*1,0})^{\odot r} \otimes E_k)$.

Definition 21. *Let τ_k be a section of (E_k, ∇_k). The pseudo-holomorphic r-jet $j^r_D\tau_k$ is a section of the bundle $\mathcal{J}^r_D E_k = (\sum_r^{j=0}(D^{*1,0})^{\odot j}) \otimes E_k$ defined out of the $r-1$-jet by the formula $j^r_D\tau_k := (j^{r-1}_D\tau_k, \partial^r_{\mathrm{sym}}\tau_k)$.*

Remark 9. *The previous definition incorporates the fact that the degree r and $r-1$ homogeneous components of the r-jet are symmetrization of the pseudo-holomorphic 1-jet of $\partial^{r-1}_{\mathrm{sym}}\tau_k$; then we have to add the homogeneous components of lower degree. Actually, we could have equally defined $j^r_D\tau_k$ by taking the symmetrization of the pseudo-holomorphic 1-jet of $j^{r-1}_D\tau_k$ (because this gives the homogeneous components of degree $1, \ldots, r$) and then adding τ_k, the degree zero homogeneous component.*

Remark 10. *The pseudo-holomorphic r-jets are useless for our purposes for low values of k. The reason is that the metric has to much influence; our only reason to use it is to be able to give a global definition of the r-jet as a section of a vector bundle, which in approximately holomorphic coordinates and using suitable trivializations of E_k is very similar to the local coupled holomorphic r-jets defined in $\mathbb{C}^n \times \mathbb{R}$ using J_0 and the flat metric (introduced in subsection 5.1); when k is large enough and for A.H sequences, due to the proximity between g_k and the flat metric, in $B(0, O(1)) \subset \mathbb{C}^n \times \mathbb{R}$ the norm of the difference (at any order) between the two notions of jet is bounded by $O(k^{-1/2})$.*

For a symplectic manifold (P, Ω) with a J-complex distribution G the bundle of pseudo-holomorphic r-jets along G will be defined to be

$$\mathcal{J}^r_G E_k := (\sum_{j=0}^{r}(G^{*1,0})^{\odot j}) \otimes E_k$$

We use the splitting $TP = G \oplus G^{\perp}$ to see $\mathcal{J}^r_G E_k$ as a subbundle of $\mathcal{J}^r E_k$; hence every section of $\mathcal{J}^r_G E_k$ can be seen as a section of $\mathcal{J}^r E_k$. We also have a natural projection $p_G \colon \mathcal{J}^r E_k \to \mathcal{J}^r_G E_k$. To define the pseudo-holomorphic r-jet along G we use the same induction procedure as in the definition of pseudo-holomorphic r-jets along D, but either before or after symmetrizing we project orthogonally $T^{*1,0}P$ over $G^{*1,0}$ (or even before taking the $(1,0)$ component we project from $T^*P_{\mathbb{C}}$ to $G^*_{\mathbb{C}}$); the result of either choice is the same.

Once approximately holomorphic coordinates have been fixed, we have a canonical pointwise $J_0 - J$ linear identification

$$\begin{array}{rcl} T\mathbb{C}^n & \to & D \\ \partial/\partial x^i_k & \mapsto & \partial/\partial x^i_k + a_i\partial/\partial s_k \\ \partial/\partial y^i_k & \mapsto & J(\partial/\partial x^i_k + a_i\partial/\partial s_k) \end{array} \tag{21}$$

The inverse of its dual is a $J_0 - J$-bundle map

$$\varpi_{k,x} \colon T^{*1,0}\mathbb{C}^n \to D^{*1,0} \tag{22}$$

It should be stressed that this identification is only important in the ball of some g_k radius O(1), the region where our computations have to be more accurate (in order to obtain local estimated transversality). There, for some constant $\gamma > 0$

$$|\varpi_{k,x}|_{g_0} \leq \gamma, \;\; |\varpi^{-1}_{k,x}|_{g_0} \leq \gamma \text{ and } |\mathrm{d}^j\varpi_{k,x}|_{g_0} \leq O(k^{-1/2}), \; \forall j \geq 1 \tag{23}$$

The gaussian decay of the reference sections will take care of what happens out of these balls. We also notice that by writing dz^i_k we will mean $\varpi_{k,x}(dz^i_k)$.

Let us assume that we have also fixed a family of reference sections of $\tau^{\mathrm{ref}}_{k,x} \in \Gamma(L^{\otimes k})$. Using any local unitary basis of E (with bounds uniform on x) together with the reference sections, we have a family of trivializations $\tau^{\mathrm{ref}}_{k,x,j}$, $j = 1, \ldots, m$, of E_k in the balls $B_{g_k}(x, \rho)$ for all x and for all k large enough. The A.H. coordinates and the associated bundle maps $\varpi_{k,x}$ provide a local basis $dz^1_k, \ldots, dz^n_k$ of $D^{*1,0}$. We obtain a family of trivializations of $\mathcal{J}^r_D E_k$ around any point as follows: for $I = (i_0, i_1, \ldots, i_n)$, with $1 \le i_0 \le m$, $0 \le i_1 + \cdots + i_n \le r$, we set

$$\mu_{k,x,I} := dz_k^{1\odot i_1} \odot \cdots \odot dz_k^{n\odot i_n} \otimes \tau^{\mathrm{ref}}_{k,x,i_0} \tag{24}$$

Definition 22. *A family of sequences $\tau_{k,x,I} \colon M \to E_k$, is called a family of holonomic frames if:*

1. *They are A.H. sections with gaussian decay w.r.t to x.*
2. *There exist $\rho, \gamma > 0$ such that in the balls $B_{g_k}(x, \rho)$ and for all point and all k large enough the sequences $j^r_D \tau_{k,x,I} \colon M \to \mathcal{J}^r_D E_k$ define a frame which is γ-comparable to $\mu_{k,x,I}$ in the following sense: if we write $j^r_D \tau_{k,x,I}$ in the basis $\mu_{k,x,I}$, for the corresponding matrix $M_{k,x}$ we have*

$$|M_{k,x}|_{g_0} \le \gamma, \ |M^{-1}_{k,x}|_{g_0} \le \gamma$$

One checks that the notion of holonomic reference frame does not depend either on the fixed approximately holomorphic coordinates, or in the chosen reference sections of E_k to define $\mu_{k,x,I}$. Only the constants involved in the definition change.

In this situation there is still a weak point. The main goal is to construct sections whose pseudo-holomorphic r-jets are transversal to certain stratifications. For that we need the pseudo-holomorphic r-jets to be A.H. sections of the bundles $\mathcal{J}^r_D E_k$ (resp. $\mathcal{J}^r E_k$ for symplectic manifolds with J-complex distribution G), so that we can apply the transversality results from approximately holomorphic theory (to be proved in section 7). We intend to use holonomic reference frames defined as follows: if I is one of the (n+1)-tuples introduced before we set

$$\nu_{k,x,I} := j^r_D \tau^{\mathrm{ref}}_{k,x,I}, \ \text{ where } \tau^{\mathrm{ref}}_{k,x,I} := (z^1_k)^{i_1} \cdots (z^n_k)^{i_n} \tau^{\mathrm{ref}}_{k,x,i_0} \in \Gamma(E_k) \tag{25}$$

In the Kähler case and due to the presence of curvature (see [5]), the coupled jets are not anymore holomorphic sections of $\mathcal{J}^r_{n,m}$ w.r.t. the complex structure induced by the connection. Similarly, the frames $\nu_{k,x,I}$ fail to be families of holonomic frames because the sections are not approximately holomorphic if $r \ge 1$.

This difficulty is overcome by introducing a new almost complex structure (a new connection) in $\mathcal{J}^r_D E_k$ (resp. $\mathcal{J}^r E_k$). This is the content of the following proposition whose proof is given in section 9.

Proposition 4. *The sequence $\mathcal{J}^r_D E_k \to (M, D, J, g_k)$ –which is very ample for the connections $\nabla_{k,r}$ previously described– admits new connections ∇_{k,H_r} such that:*

1. *$\nabla_{k,r} - \nabla_{k,H_r} \in D^{*0,1} \otimes \mathrm{End}(\mathcal{J}^r_D E_k)$. Hence, if in order to compute the pseudo-holomorphic jets (definition 21) we use the connections ∇_{k,H_r} instead of $\nabla_{k,r}$, then the result is the same.*
2. *Let us denote the curvatures of ∇_{k,H_r} and $\nabla_{k,r}$ by F_{k,H_r} and $F_{k,r}$ respectively. Then $F_{k,H_r} \approxeq F_{k,r}$ and hence $(\mathcal{J}^r_D E_k, \nabla_{k,H_r})$ is a very ample sequence.*
3. *If $\tau_k \colon M \to E_k$ is a C^{r+h}-A.H. sequence of sections, $j^r_D \tau_k \colon M \to \mathcal{J}^r_D E_k$ is a C^h-A.H. sequence of sections for the connections ∇_{k,H_r}.*

In the integrable model $(E, \nabla) \to (\mathbb{C}^n \times \mathbb{R}, D_h, J_0, g_0)$, with $E = L_1 \oplus \cdots \oplus L_m$, if the curvature F_i of each line bundle L_i, $i = 1, \ldots, m$, restricted to the leaves is of type $(1,1)$ and has constant components w.r.t. the coordinates $z_1, \ldots, z_n$, then the restrictions to each leaf of the curvatures F_{H_r} and F_r (point 2 above) coincide. As

a consequence the new almost CR structure in the total space of $\mathcal{J}^r_{D_h,n,m}$ induced by ∇_{H_r} is also integrable (the foliation does not vary, just the leafwise complex structure). Also if τ is a CR section ($\mathbb{C}^m$-valued function), then the coupled CR jet $j^r_{CR}\tau$ is a CR section of $(\mathcal{J}^r_{D_h,n,m}, \nabla_{H_r})$.

In the case of (P,Ω) symplectic with a J-complex distribution G, analogous results hold for $\mathcal{J}^r E_k$ and for the integrable model.

Proof. As we said we postpone the proof until section 9, but we introduce the formula for the connection.

Let $\sigma_k = (\sigma_{k,0}, \sigma_{k,1})$ be a section (maybe local) of $\mathcal{J}^1_D E_k$. We define

$$\nabla_{H_1}(\sigma_{k,0}, \sigma_{k,1}) = (\nabla\sigma_{k,0}, \nabla\sigma_{k,1}) + (0, -F^{1,1}_D \sigma_{k,0}),$$

where $F^{1,1}_D \sigma_{k,0} \in D^{*0,1} \otimes D^{*1,0} \otimes E_k$ (see [5]). $\square$

Remark 11. *The approximate equality $F_{H_1,k} \approxeq F_k$ has interesting consequences. If for simplicity $E_k = L^{\otimes k}$, in approximately holomorphic coordinates after the local identification of $D^{*1,0}$ with $T^{*1,0}\mathbb{C}^n$ and for choices of trivializations τ_k whose associated connection form is A in equation 3, in the basis $(1,0)\otimes\tau_k, (0,dz^1_k)\otimes\tau_k, \ldots, (0,dz^n_k)\otimes\tau_k$, the connection matrix of ∇_{k,H_1} in $B(0,O(1)) \subset \mathbb{C}^n \times \mathbb{R}$ is, up to summands bounded (at any order) by $O(k^{-1/2})$*

$$\begin{vmatrix} A & -\frac{1}{2}d\bar{z}^1_k & \cdots & -\frac{1}{2}d\bar{z}^n_k \\ 0 & A & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & A \end{vmatrix}$$

In particular we have a uniform control on the new metric of the total space of the bundles $\mathcal{J}^1_D L_k$ (resp. $\mathcal{J}^1 L_k$). In a similar manner this uniform control also holds for the bundles $\mathcal{J}^r_D E_k$ (resp. $\mathcal{J}^r E_k$).

The above property will imply that if we have a sequence of stratifications $\mathcal{S}$ such that for a choice of approximate holomorphic coordinates and reference frames, in the associated local basis $\mu_{k,x,I}$ of equation 24 the strata S^a_k are given by equations (functions) that do not depend neither on k nor on x, then the different bounds associated to the strata (basically those of the local functions defining them) will not depend on k and x (we can compute them for the corresponding model with the Euclidean metric elements).

6. The linearized Thom-Boardman stratification

For the very ample sequences E_k there is an easy sufficient condition for a sequence of stratifications to be finite, Whitney and approximately holomorphic.

Let us denote by T the group of translations of $\mathbb{C}^n \times \mathbb{R}$ (resp. $\mathbb{C}^p$ in the relative case).

Lemma 13. *Let $(S^a_k)_{a\in A}$ be a sequence of stratifications of $E_k \to (M, D, \omega)$ such that for a choice of approximately holomorphic coordinates and approximately holomorphic trivialization it is sent to $S^a{}_{a\in A}$, a CR finite Whitney stratification of $\underline{\mathbb{C}}^m \to \mathbb{C}^n \times \mathbb{R}$ transversal to the fibers. Then the sequence $(S^a_k)_{a\in A}$ is as in definition 18.*

From a Whitney CR stratification of $\underline{\mathbb{C}}^m \to \mathbb{C}^n \times \mathbb{R}$ transversal to the fibers and invariant under the action of $\mathsf{T}\times Gl(m,\mathbb{C})$ (or $\mathsf{T}\times\mathbb{C}^*$), using the local identifications of E_k with $\underline{\mathbb{C}}^m$ furnished by A.H. coordinates and A.H. trivializations it is possible to induce an approximately holomorphic sequence of finite Whitney stratifications of E_k.

Proof. Recall that we are interested in constructing A.H. sequences of sections transversal to $(S_k^a)_{a\in A}$; in particular this sections will be uniformly bounded. Therefore, for each k, x we can work in the subset $B(0, O(1))\times B(0, R) \subset (\mathbb{C}^n\times\mathbb{R})\times\mathbb{C}^m = \underline{\mathbb{C}}^m$, for some $R > 0$. Let f be a function defining locally a stratum S, which by hypothesis can be chosen to be CR. Condition 1 in definition 18 holds trivially for the model S and therefore for $(S_k^a)_{a\in A}$, because when we compare the Euclidean metric and $\hat{g}_k$ we get the same inequalities as in condition 1 in definition 8.

Since the model stratification is Whitney *and we work in a compact region* the Whitney condition implies the estimated Whitney condition for the Euclidean metric and hence for $\hat{g}_k$.

Let $\hat{J}_0$ be the leafwise holomorphic structure associated to the canonical CR structure of $\underline{\mathbb{C}}^m = (\mathbb{C}^n \times \mathbb{R}) \times \mathbb{C}^m$ and let $\hat{D}_h$ denote the foliation by complex hyperplanes. Since the local function f defining S is CR, in particular it is fiberwise holomorphic, and this proves point 2 in definition 18.

Let $(\hat{D}, \hat{J}, \hat{g}_k)$ be the almost CR structure on $B(0, O(1)) \times B(0, R)$ induced by the one on E_k. In order to prove point 3 it suffices to check that f is A.H. w.r.t. the this almost CR structure. We are going to slightly modify the induced almost CR structure: instead of $\hat{D}$, we select $\hat{D}_h$. By using the Euclidean orthogonal projection, we can push $\hat{J}\colon \hat{D} \to \hat{D}$ into an almost complex structure $J'\colon \hat{D}_h \to \hat{D}_h$.

Since $|\mathrm{d}^j(\hat{D} - \hat{D}_h)|_{g_0} \le O(k^{-1/2})$, for all $j \ge 0$, then f is A.H. w.r.t. $(\hat{D}, \hat{J}, \hat{g}_k)$ if and only if it is A.H. w.r.t. $(\hat{D}_h, J', g_0)$ (this appears in the proof of lemma 3).

In $\underline{\mathbb{C}}^m = (\mathbb{C}^n \times \mathbb{R}) \times \mathbb{C}^m$ we have canonical coordinates $z_k^1, \dots, z_k^n, s_k, u_k^1, \dots, u_k^m$. These are CR coordinates w.r.t $(\hat{D}_h, \hat{J}_0)$. By hypothesis

$$\frac{\partial f}{\partial \bar{z}_k^1} = \cdots \frac{\partial f}{\partial \bar{z}_n^1} = \frac{\partial f}{\partial \bar{u}_k^1} = \cdots \frac{\partial f}{\partial \bar{u}_k^m} = 0$$

If we show that $z_k^1, \dots, z_k^n, s_k, u_k^1, \dots, u_k^m$ are A.H. coordinates for $(\hat{D}_h, J', g_0)$ then we are done (this is again lemma 3 in the absence of connection form). But this follows from the fact that the trivialization of $\underline{\mathbb{C}}^m$ is given by an A.H. frame and therefore the induced distribution (by the connection form) $\mathcal{H}$ on $\hat{D}_h$ is such that $|\mathrm{d}^j(\mathcal{H} - \hat{J}_0\mathcal{H})|_{g_0} \le O(k^{-1/2})$, for all $j \ge 0$.

To prove the result in the other direction, we fix A.H. coordinates and A.H. frames for E_k. The $\mathsf{T} \times Gl(m, \mathbb{C})$-invariance of ${S^a}_{a\in A} \subset \underline{\mathbb{C}}^m$ means that the local identifications define a sequence of global stratifications, and that these do not depend either on the A.H. coordinates or on the A.H. trivializations. It is an approximately holomorphic sequence of finite Whitney stratifications by the first part of the proof.

□

In contrast to what happens for 0-jets, it is not easy to find non-trivial approximately holomorphic stratifications for higher order jets. The difficulty comes from the fact that the modification of the connection of proposition 4 that makes the r-jets of A.H. sequences of sections of E_k into A.H. sequences of sections of $\mathcal{J}_D^r E_k$, makes it very complicated to guarantee that the strata are given by functions whose composition with an A.H. section is an A.H. function.

Example 1. *Let $L_\Omega^{\otimes k}$ be the sequence of powers of the pre-quantum line bundle of a* symplectic manifold*. Let us consider the following sequence of strata in $\mathcal{J}^1 L_\Omega^{\otimes k}$:*

$$\Sigma_{k,p} = \{(\sigma_0, \sigma_1) | \sigma_1 = 0\}$$

Using the base $\mu_{k,x,I}$ of equation 24, where $I = 1, \dots, p$, and taking reference sections in Darboux charts, we get coordinates $z_k^1, \dots, z_k^p, v_k^0, v_k^1, \dots, v_k^p$ for the total space. $\Sigma_{k,p}$ is then defined by the zeros of the function $f = (v_k^1, \dots, v_k^p)\colon \mathbb{C}^{2p+1} \to$

$\mathbb{C}^p$, which is not holomorphic (or A.H.) w.r.t. the modified almost complex structure of the total space. Otherwise, the composition $f \circ j^1(z_k^1 \tau_{k,x}^{\mathrm{ref}})$ would be A.H., but that composition is $(1 + z_k^1 \bar{z}_k^1, z_k^1 \bar{z}_k^2, \ldots, z_k^1 \bar{z}_k^p)$.

Actually, we cannot find A.H. functions f defining $\Sigma_{k,p}$: let us work in Darboux coordinates with the canonical complex structure J_0 in the base. Assume that $\mu_{k,x,I}$ is built out of the reference section $\mathrm{e}^{-|z_k|^2/4}\xi$, where ξ is a unitary trivialization of L_Ω whose connection form is A. Then $\mathcal{J}^1 L_\Omega^{\otimes k}$ becomes locally $\underline{\mathbb{C}}^{p+1}$ with diagonal connection matrix $A\mathrm{I}_{p+1\times p+1}$. Proposition 4 for complex manifolds implies that the modified almost complex structure on $\underline{\mathbb{C}}^{p+1}$ is integrable. The submanifold $z_k^2 = \cdots = z_k^p = v_k^2 = \cdots = v_k^p = 0$ is complex w.r.t. the modified almost complex structure. Therefore, we can restrict our attention to the case $p = 1$. The sections $j^1_{\mathrm{hol}}\mathrm{e}^{-|z_k|^2/4}\xi$, $j^1_{\mathrm{hol}} z_k \mathrm{e}^{-|z_k|^2/4}\xi$ are by proposition 4 holomorphic. If we use it to trivialize $\mathcal{J}^1 L_\Omega^{\otimes k}$ in a neighborhood of the origin, then we obtain a new identification with $\underline{\mathbb{C}}^3$ with its canonical complex structure. *Let z_k,t_k,s_k be the new complex coordinates. A short computation shows that*

$$\begin{aligned} v_k^0 &= t_k + z_k s_k \\ v_k^1 &= -\bar{z}_k/2 t_k + (1 - z_k \bar{z}_k/2) s_k \end{aligned}$$

Hence away from the origin $\Sigma_{k,p}$ admits the parametrization

$$(z_k, s_k) \mapsto (z_k, s_k, s_k(2/\bar{z}_k - z_k))$$

Therefore, $\Sigma_{k,p}$ is not holomorphic w.r.t. the modified almost complex structure, and it follows that we cannot find f A.H. defining $\Sigma_{k,p}$ locally.

6.1. **Quasi-stratifications.** For the applications we have in mind the notion of stratification has to be weakened. We start doing it for the local model (endowed with the trivial connection).

Let $\sigma \in S \subset \mathcal{J}^{r+1}_{D_h,n,m}$. We say that $\alpha \in \Gamma(\mathcal{J}^r_{D_h,n,m})$ is a *local representation* for σ if $\alpha(0) = \pi_r^{r+1}\sigma$ ($\pi_r^{r+1}\colon \mathcal{J}^{r+1}_{D_h,n,m} \to \mathcal{J}^r_{D_h,n,m}$ the natural projection) and $\sigma = j^1_{D_h}\alpha(0) \in \mathcal{J}^{r+1}_{D_h,n,m}$, where $j^1_{D_h}\alpha$ (or $j^1_{CR}\alpha$ if there is no possible confusion about the CR structure) denotes the CR 1-jet of α. The last equality should be understood in the following sense: the degree 1-component of the 1-jet should give an element of $\mathcal{J}^{r+1}_{D_h,n,m}$ (with vanishing degree 0 homogeneous component) and whose homogeneous components of degree $1, \ldots, r+1$ coincide with those of σ.

Definition 23. *(see* [5]*) Let S be a submanifold of $\mathcal{J}^r_{D_h,n,m}$ (resp. $\mathcal{J}^r_{\mathbb{C}^g,p,m}$). We define Θ_S to be the set of points $\sigma \in S$ for which there exists an (r+1)-jet $\tilde{\sigma}$ (resp. (r+1)-jet along G) such that $\pi_r^{r+1}\tilde{\sigma} = \sigma$ and with a local representation α intersecting S in σ transversely along D_h (resp. along $\mathbb{C}^g$). We refer to Θ_S as the holonomic transversal subset of S.*

It can be checked that if S is invariant the action of $\mathsf{T} \times (Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C}))$ -the second factor $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C})$ acting fiberwise- (resp. $\mathsf{T} \times (Gl(g,\mathbb{C}) \times Gl(m,\mathbb{C})))$, then Θ_S has the same invariance property.

When an (r+1)-jet σ is represented by a local section of $\mathcal{J}^r_{D_h,n,m}$, in order to check whether $\pi_r^{r+1}\sigma \in S$ belongs to Θ_S the local representation is essentially unique: regarding transversality, it is enough to consider the degree 1 part of the Taylor expansion in the coordinates $z_k^1, \bar{z}_k^1, \ldots, z_k^n, \bar{z}_k^n$ (we turn the section into a function using the basis μ_I). The degree 0 part is determined by the r-jet, the hypothesis imply that the antiholomorphic part is vanishing and the holomorphic part is determined by the (r+1)-jet. That means in particular that we can restrict our attention to CR representations if necessary.

The importance of Θ_S is two-fold: on the one hand it will be used to define the stratifications we are interested in. On the other hand it is a very relevant subset when we study transversality to the strata: indeed, if τ is a CR section of $\underline{\mathbb{C}}^m$ and $\alpha := j^r_{D_h}\tau$ is such that $\alpha(0) = \sigma$ and $\sigma \notin \Theta_S$, then α cannot be transversal along D_h to S at σ (notice that $\tilde{\sigma} := (\tau(0), \mathrm{d}_{D_h}\alpha(0)) = j^{r+1}_{D_h}\tau(0) \in \mathcal{J}^{r+1}_{D_h,n,m}$, and therefore α is a local representation of $\tilde{\sigma}$). The consequence is that if $S\backslash\Theta_S \subset \partial S'$, transversality of τ to S implies that τ misses a neighborhood of $S\backslash\Theta_S$ in S'.

Definition 23 extends to strata $S_k \subset \mathcal{J}^r_D E_k$ (resp. $\mathcal{J}^r_G E_k$): we have a notion of pseudo-holomorphic 1-jet of a section of $\mathcal{J}^r_D E_k$ (resp. pseudo-holomorphic 1-jet along G of a section of $\mathcal{J}^r_G E_k$) -because we have a connection $\nabla_{H,D}$ (resp. a connection on $\mathcal{J}^r_G E_k$ defined out of ∇_H and the projection $p_G\colon \mathcal{J}^r E_k \to \mathcal{J}^r_G E_k$)- and hence the notion of local representation. Then Θ_{S_k} are those points σ with lifts $\tilde{\sigma}$ having a local representation transversal to S_k at σ along D (resp. G).

Recall that associated to a family of A.H. charts we have identifications $\varpi_{k,x}\colon T^{*1,0}\mathbb{C}^n \to D^{*1,0}$. If we also fix a family of A.H. trivializations of E_k over the charts there is an induced identification

$$\Pi_{k,x}\colon \mathcal{J}^r_D E_k \to \mathcal{J}^r_{D_h,n,m} \tag{26}$$

Lemma 14. *Let S_k be a sequence of strata of $\mathcal{J}^r_D E_k$, where either $r = 0$ and $E_k = E \otimes L^{\otimes k}$, or $E_k = \underline{\mathbb{C}}^m$ and $r \in \mathbb{N}$.*

1. *If $E_k = \underline{\mathbb{C}}^m$ assume that for a choice of A.H. charts $\Pi_{k,x}(S_k) = S$, where $S \subset \mathcal{J}^r_{D_h,n,m}$ is invariant under the action $\mathsf{T} \times Gl(n,\mathbb{C})$. Then $\Pi_{k,x}(\Theta_{S_k}) = \Theta_S$.*
2. *The same result holds for $E \otimes L^{\otimes k}$ and $r = 0$; we need to fix A.H. trivializations of E_k (so $\Pi_{k,x}$ is defined) and require invariance of $S \subset \underline{\mathbb{C}}^m$ under the action of $\mathsf{T} \times Gl(m,\mathbb{C})$.*

For jets along G we have analogous results, but we need A.H. charts adapted to G and we ask for $\mathsf{T} \times (Gl(g,\mathbb{C}))$-invariance of S instead of for $\mathsf{T} \times Gl(n,\mathbb{C})$-invariance.

Proof. Since S is $Gl(n,\mathbb{C})$-invariant, so is Θ_S.

We have the local identifications $\Pi_{k,x}\colon \mathcal{J}^r_D E_k \to \mathcal{J}^r_{D_h,n,m}$.

Let $y \in M$ belong to $B(0, O(1))$ in the domain of the charts centered at x_1 and x_2, for some k. Then there is a fiber bundle isomorphism

$$\Phi_{k,x_1,x_2}\colon \mathcal{J}^r_{D_h,n,m} \to \mathcal{J}^r_{D_h,n,m} \tag{27}$$

defined as follows: for each point y in the intersection of the domains of the charts, the restriction of the differential to D is a complex J-linear map L_y. Consider the linear map $\varpi_{k,x_2} \circ L^*_y \circ \varpi^{-1}_{k,x_2}\colon T^{*1,0}\mathbb{C}^n \to T^{*1,0}\mathbb{C}^n$, which belongs to $Gl(n,\mathbb{C})$. Φ_{k,x_1,x_2} in the fiber over y (or over the origin in both charts due to the T-invariance) is the vector space isomorphism induced by $\varpi_{k,x_2} \circ L^*_y \circ \varpi^{-1}_{k,x_2}$ (and the identity acting on the $\mathbb{C}^m$ factor of the tensor product).

Since S is invariant under the $\mathsf{T} \times Gl(n,\mathbb{C})$-action, $\Phi_{k,x_1,x_2}(\Theta_S, S) = (\Theta_S, S)$. In particular the pair (Θ_S, S) does not depend on the chosen family of A.H. charts. We construct an appropriate family of A.H. charts (there is no Darboux condition involved here) by the usual rescaling procedure, but starting from normal coordinates composed with a linear transformation so that $(D, J) = (D_h, J_0)$ at the origin. Recall that since $E_k = \underline{\mathbb{C}}^m$, the connection $\nabla_{k,r}$ on $\mathcal{J}^r_D E_k$ is just induced by the Levi-Civita connection (in the $\underline{\mathbb{C}}^m$ factor we use the trivial one d). Hence the pushforward of $\nabla_{k,r}$ by $\Pi_{k,x}$ to $\mathcal{J}^r_{D_h,n,m}$ has vanishing connection form at the origin. Since we also have $(D \oplus D^\perp, J) = (D_h \oplus D_v, J_0)$, for any section α of $\mathcal{J}^r_{D_h,n,m}$ we have $j^1_D\alpha(0) = j^1_{D_h}\alpha(0)$. Therefore, the local representations at the origin for

the canonical CR structure and the induced one coincide. From that and $D = D_h$ at the origin we conclude $\Pi_{k,x}(\Theta_{S_k}) = \Theta_S$.

Point 2 is proven in the same fashion. The $Gl(m,\mathbb{C})$-invariance implies that we can choose any arbitrary family of A.H. trivializations. What we do is selecting trivializations such that the connection form over the origin is vanishing (here we deal with the connection ∇_k on E_k).

Notice that we cannot state point 2 for higher order jets because we do not whether the action of $Gl(n,\mathbb{C}) \times Gl(m,\mathbb{C})$ allows us to kill at the origin of each chart the connection form of the modified connection ∇_{k,H_r}.

For the relative results we start by modifying a bit the vector bundle isomorphism $\varpi_{k,x} \colon T^{*1,0}\mathbb{C}^p \to T^{*1,0}P$; the original $J_0 - J$-complex map $T\mathbb{C}^p \to TP$ can be easily arranged to be compatible with the splittings $T\mathbb{C}^g \oplus T\mathbb{C}^{p-g}$ and $G \oplus G^\perp$. Due to the $\mathsf{T} \times (Gl(g,\mathbb{C}))$-invariance we are free to pick any family of A.H. charts adapted to G. The ones we need come from rescaling normal coordinates composed with a linear transformation sending $(G \oplus G^\perp, J)$ to $(\mathbb{C}^g \oplus \mathbb{C}^{p-g}, J_0)$ at the origin. In this coordinates the connection form on $T^{*1,0}\mathbb{C}^g$ is vanishing, because we project the Levi-Civita connection which is already vanishing at the origin. Hence the 1-jets along G and $\mathbb{C}^g$ at the origin coincide (also because of the equality $(G \oplus G^\perp, J)$ to $(\mathbb{C}^g \oplus \mathbb{C}^{p-g}, J_0)$ at the origin), and this proves the result. □

The only relevant strata $S_k \subset \mathcal{J}^r_D E_k$ for which we have to consider the subsets Θ_{S_k} are the zero sections Z_k. In that case (see [3]) the subsets Θ_{Z_k} are those r-jets whose degree 1 component is onto.

Definition 24. *An approximately holomorphic quasi-stratification of $\mathcal{J}^r_D E_k$ is an approximately holomorphic stratification in which if a strata $S_k \neq Z_k$ approaches Z_k, then it accumulates into points of $Z_k \backslash \Theta_{Z_k}$.*

6.2. **The Thom-Boardman-Auroux stratification for maps to projective spaces.** Let $E_k = \underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$. Let $Z^0, \ldots, Z^m$ be the complex coordinates associated to the trivialization of $\mathbb{C}^{m+1}$ (at any fiber) and let $\pi \colon \mathbb{C}^{m+1}\backslash\{0\} \to \mathbb{CP}^m$ be the canonical projection. Consider the canonical affine charts

$$\begin{array}{rcl} \varphi_i^{-1} \colon U_i & \longrightarrow & \mathbb{C}^m \\ [Z_0 : \cdots : Z_m] & \longmapsto & \left(\dfrac{Z^1}{Z^0}, \ldots, \dfrac{Z^{i-1}}{Z^0}, \dfrac{Z^{i+1}}{Z^0}, \ldots, \dfrac{Z^m}{Z^0} \right) \end{array}$$

For each chart φ_i we consider the bundle

$$\mathcal{J}^r_D(M,\mathbb{C}^m)_i := (\sum_{j=0}^{r} (D^{*1,0})^{\odot j}) \otimes \underline{\mathbb{C}}^m \tag{28}$$

We now bring back the discussion at the beginning of section 5. Assume for the moment that M is a Levi-flat CR manifold and fix a family of CR charts. Over each of the balls $B_{g_k}(x, O(1))$ we have the bundles $\mathcal{J}^r_{D_h,n,m}$ of CR r-jets. Notice that if we use the frames $\mu_{k,x,I}$ of equation 24 they are vector bundles.

The local bundles $\mathcal{J}^r_{D_h,n,m}$ glue into the non-linear bundle $\mathcal{J}^r_{\mathcal{D}}(M,\mathbb{C}^m)_i$ (we will also use the notation $\mathcal{J}^r_{CR}(M,\mathbb{C}^m)_i$): let $y \in M$ be a point belonging to two different charts centered at x_0 and x_1 respectively. If we send y in both charts to the origin via a translation, then the change of coordinates restricts to the leaf through the origin to a holomorphic map fixing the origin. The fibers over y are related by the action of the holomorphic r-jet of the bi-holomorphism. If we only take the linear part of the action –which is the vector bundle map Φ_{k,x_1,x_2} of equation 27– we are equally defining a bundle, for the cocycle condition still holds. Moreover, it is a vector bundle. Besides, since we only use the linear part we do not need either D

or J to be integrable. This bundle is $\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ as defined in equation 28 (what we defined there is rather a sequence in which the metric in the $D^{*1,0}$ factors is induced from g_k). Thus for Levi-flat manifolds the vector bundles $\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ are "linear approximations" of the non-linear bundles $\mathcal{J}^r_{CR}(M,\mathbb{C}^m)_i$.

Proposition 5.

(1) *The vector bundles $\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ can be glued to define the almost complex fiber bundles $\mathcal{J}^r_D(M,\mathbb{CP}^m)$ of pseudo-holomorphic r-jets of maps from M to $\mathbb{CP}^m$, so that their fibers inherit a canonical holomorphic structure.*

(2) *Given $\phi_k\colon M\to\mathbb{CP}^m$ there is a notion of pseudo-holomorphic r-jet extension $j^r_D\phi_k\colon M\to\mathcal{J}^r_D(M,\mathbb{CP}^m)$, which is compatible with the notion of pseudo-holomorphic r-jet for the sections $\varphi_i^{-1}\circ\phi_k\colon M\to\mathbb{C}^m$ of definition 21. If $\phi_k\colon M\to\mathbb{CP}^m$ is an A.H. sequence then $j^r_D\phi_k\colon M\to\mathcal{J}^r_D(M,\mathbb{CP}^m)$ is also A.H.*

Analogous results hold in the relative setting for the bundles $\mathcal{J}^r(P,\mathbb{CP}^m)$ and $\mathcal{J}^r_G(P,\mathbb{CP}^m)$. Also there is an approximately holomorphic sequence of canonical submersions $p_G\colon\mathcal{J}^r(P,\mathbb{CP}^m)\to\mathcal{J}^r_G(P,\mathbb{CP}^m)$. These submersions are left inverses of the natural inclusions $l_G\colon\mathcal{J}^r_G(P,\mathbb{CP}^m)\hookrightarrow\mathcal{J}^r(P,\mathbb{CP}^m)$ so that for $\phi_k\colon P\to\mathbb{CP}^m$ an A.H. sequence, $j^r_G\phi_k\colon P\to\mathcal{J}^r_G(P,\mathbb{CP}^m)\hookrightarrow\mathcal{J}^r(P,\mathbb{CP}^m)$ is A.H.

Proof. Let us denote the change of coordinates $\varphi_j^{-1}\circ\varphi_i$ by Ψ_{ji}.

For any $y\in M$ the points in $\{y\}\times(U_i\cap U_j)\subset\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ are identified with points in $\{y\}\times(U_i\cap U_j)\subset\mathcal{J}^r_D(M,\mathbb{C}^m)_j$ using the same transformation $j^r\Psi_{ji}$ in $\mathcal{J}^r_{n,m}$ induced by the holomorphic change of coordinates Ψ_{ji}. In other words, if we take an approximately holomorphic chart centered at x say and containing y, we get as in equation 26 a vector bundle isomorphism $\Pi_{k,x,i}\colon\mathcal{J}^r_D(M,\mathbb{C}^m)_i\to\mathcal{J}^r_{D_h,n,m}$. Thus for $\sigma\in\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ there exists $F\colon\mathbb{C}^n\to\mathbb{C}^m$ a CR function such that $\Pi_{k,x,i}(\sigma)=j^r_{D_h}F(x)$.

The bundle map we define is:

$$\begin{array}{rcl} j^r\Psi_{ji}\colon\mathcal{J}^r_D(M,\mathbb{C}^m)_i & \longrightarrow & \mathcal{J}^r_D(M,\mathbb{C}^m)_j \\ \sigma & \longmapsto & \Pi^{-1}_{k,x,j}(j^r_{D_h}(\Psi_{ji}\circ F)(x)) \end{array} \tag{29}$$

This map does not depend either on the charts: if we have a point y in two different charts centered at x_1 and x_2, then we saw in the proof of lemma 14 that the vector space isomorphism $\Phi_{k,x_1,x_2}\colon\mathcal{J}^r_{D_h,n,m}\to\mathcal{J}^r_{D_h,n,m}$ was induced by $T\in Gl(n,\mathbb{C})$. The bundle map of equation 29 is equivariant w.r.t this action, because in the CR setting it is equivariant w.r.t the action in the base of CR transformations. Hence, the result follows by considering the affine CR transformation sending y in the first chart to its image in the second and whose linear part is $T^*\times\mathrm{I}\colon\mathbb{C}^n\times\mathbb{R}\to\mathbb{C}^n\times\mathbb{R}$.

Equivalently, the r-jet of $\Psi_{ji}\circ F$ admits a coordinate free expression only in terms the r-jet of F.

Remark 12. *If our manifold is CR and we have x belonging to two different CR charts, then there is a natural induced identification $\mathcal{J}^r_{D_h,n,m}\to\mathcal{J}^r_{D_h,n,m}$ over the points belonging to both charts. This identification is just the action of the CR r-jet of the change of coordinates. We observe that this is not the action of Φ_{k,x_1,x_2}, with is just the action of the 1-jet of the change of coordinates (the only one available for almost CR structures).*

Therefore the identifications $j^r\Psi_{ji}$ give rise to a well defined locally trivial fiber bundle $\mathcal{J}^r_D(M,\mathbb{CP}^m)$.

The fibers of $\mathcal{J}^r_D(M,\mathbb{CP}^m)$ admit a canonical holomorphic structure because using the local identifications $\Pi_{k,x,i}$ the fiber is some $\mathbb{C}^N$ and the change of coordinates is a holomorphic map (because it is the holomorphic r-jet of Ψ_{ji}), and this proves part 1.

Let $\phi\colon (M,J,D)\to\mathbb{CP}^m$. Its pseudo-holomorphic r-jet $j^r_D\phi$ is defined as follows: the affine charts of projective space induce maps $\phi_i := \varphi_i^{-1}\circ\phi\colon M\to\mathbb{C}^m$. Using the trivial connection d in this trivial vector bundle and the induced connection on $D^{*1,0}$, we can define the corresponding pseudo-holomorphic r-jet $j^r_D\phi_i$ (definition 21). We must check that $j^r_D\phi_j = j^r\Psi_{ji}(j^r_D\phi_i)$.

More generally instead of using a holomorphic diffeomorphism $\Psi_{ji}\colon\mathbb{C}^m\to\mathbb{C}^m$, we consider any holomorphic map $H\colon\mathbb{C}^{m_1}\to\mathbb{C}^{m_2}$ and use the local identifications $\Pi_{k,x,s}\colon\mathcal{J}^r_D(M,\mathbb{C}^{m_s})\to\mathcal{J}^r_{D_h,n,m_s}$, $s=1,2$, to induce a map $j^rH\colon\mathcal{J}^r_D(M,\mathbb{C}^{m_1})\to\mathcal{J}^r_D(M,\mathbb{C}^{m_2})$, so that for a function $\phi\colon M\to\mathbb{C}^{m_1}$ the equation $j^r_D(H\circ\phi) = j^rH(j^r_D\phi)$ holds.

The proof is by induction on r. First we may assume $m_2=1$ and it is enough to check the equality for the degree r homogeneous component of the r-jet. We shall denote the degree r-homogeneous component of j^rH by d^rH; recall that $\mathrm{d}^rH(j^r_D\phi(x))$ depends on the components of every order of $j^r_D\phi(x)$. Let $F=(F^1,\dots,F^{m_1})\colon\mathbb{C}^n\times\mathbb{R}\to\mathbb{C}^{m_1}$ be a CR function such that

$$j^r_D\phi(x) = j^r_{D_h}F(x)$$

Also the degree r-homogeneous component of $j^r_{D_h}F$ is denoted by ∂^r_0F. By definition

$$\partial^j_{\mathrm{sym}}\phi(x) = \partial^j_0F(x),\ j=0,\dots,r \tag{30}$$

Once we use the identification $\partial\phi(x)=\partial_0F(x)$, we have

$$\mathrm{d}H(\partial\phi(x)) := \mathrm{d}H(\partial_0F(x)) = \sum_{a=1}^{m_1}\frac{\partial_0H}{\partial_0z^a}\partial_0F^a(x), \tag{31}$$

and using the identification of equation 30 back we conclude

$$\mathrm{d}H(\partial\phi(x)) = \mathrm{d}H(\partial_0F(x)) = \sum_{a=1}^{m_1}\frac{\partial_0H}{\partial_0z^a}\partial\phi^a(x) \tag{32}$$

The partial derivatives of H are evaluated on $\phi(x)=F(x)$, but we omit it in the notation.

The computation of $\partial(H\circ\phi)(x)$ is done by firstly taking in $\nabla(H\circ\phi)(x)$ its projection over D^* (or restricting the differential to D). Since

$$\nabla(H\circ\phi)(x) = \sum_{a=1}^{m_1}\frac{\partial_0H}{\partial_0z^a}\nabla\phi^a(x) \tag{33}$$

is the sum of partial derivatives of H multiplied by the components $\nabla\phi_a(x)$ of $\nabla\phi(x)$, taking $\nabla_D(H\circ\phi)(x)$ amounts to substituting in equation 33 the factors $\nabla\phi^a(x)$ by $\nabla_D\phi^a(x)$.

Next the holomorphic component is singled out; since H is holomorphic, $\partial(H\circ\phi)(x)$ is computed by taking the component $\partial\phi^a(x)$ of $\nabla_D\phi^a(x)$ in equation 33. Thus we obtain the same result as in equation 32.

Regarding the 2-jets,

$$\mathrm{d}^2H(j^2_{D_h}F(x)) = \sum_{b,a=1}^{m_1}\frac{\partial_0^2H}{\partial_0z^a\partial_0z^b}\partial_0F^a(x)\otimes\partial_0F^b(x) + \sum_{c=1}^{m_1}\frac{\partial_0H}{\partial_0z^c}\partial_0^2F^c(x), \tag{34}$$

so using equation 30 we have:

$$\mathrm{d}^2 H(j^2_D\phi(x)) = \sum_{b,a=1}^{m_1} \frac{\partial_0^2 H}{\partial_0 z^a \partial_0 z^b}\partial\phi^a(x)\otimes\partial\phi^b(x) + \sum_{c=1}^{m_1}\frac{\partial_0 H}{\partial_0 z^c}\partial^2_{\mathrm{sym}}\phi^c(x) \tag{35}$$

To compute $\partial^2_{\mathrm{sym}}(H\circ\phi)(x)$ we first differentiate $\partial(H\circ\phi)$ at x:

$$\nabla\partial(H\circ\phi)(x) = \sum_{b,a=1}^{m_1} \frac{\partial_0^2 H}{\partial_0 z^a \partial_0 z^b}\nabla\phi^a(x)\otimes\partial\phi^b(x) + \sum_{c=1}^{m_1}\frac{\partial_0 H}{\partial_0 z^c}\nabla\partial\phi^c(x) \tag{36}$$

Taking the component along D and then the holomorphic part amounts to substituting in equation 36 $\nabla\phi^a(x)$ by $\partial\phi^a(x)$ and $\nabla\partial\phi^c(x)$ by $\partial^2\phi^c(x)$:

$$\partial^2(H\circ\phi)(x) = \sum_{b,a=1}^{m_1} \frac{\partial_0^2 H}{\partial_0 z^a \partial_0 z^b}\partial\phi^a(x)\otimes\partial\phi^b(x) + \sum_{c=1}^{m_1}\frac{\partial_0 H}{\partial_0 z^c}\partial^2\phi^c(x) \tag{37}$$

We need to show that symmetrizing equation 37 amounts to writing $\partial^2_{\mathrm{sym}}\phi^c(x)$ instead of $\partial^2\phi^c(x)$.

In equation 37 we have terms of “type” 2 –those containing a second derivative of ϕ– and terms of “type” $(1,1)$ which contain the tensor product of two derivatives of ϕ. Terms of “type” $(1,1)$ are already symmetric (just exchange the indices a,b); the symmetrization –being a linear projection– does not alter them. Now one checks that the symmetrization of each summand $\frac{\partial_0 H}{\partial_0 z^c}\partial^2\phi^c(x)$ is exactly $\frac{\partial_0 H}{\partial_0 z^c}\partial^2_{\mathrm{sym}}\phi^c(x)$.

By induction we assume $\mathrm{d}^r H(j^r_D\phi(x)) = \partial^r_{\mathrm{sym}}(H\circ\phi)(x)$.

By a partition of r of degree s we understand any (ordered) s-tuple $(r_1,\dots,r_s)$, $1\le s\le r$, $1\le r_j\le r$, $r_1+\dots+r_s=r$. In the computation of $\mathrm{d}^r H(j^r_D\phi(x)) := \partial_0^r(H\circ F)(x)$ we get an algebraic expression whose summands are of the form

$$\frac{\partial_0^{r_1+\dots+r_s}H}{\partial_0^{r_1}z^{i_1}\dots\partial_0^{r_s}z^{i_s}}\partial_0^{r_1}F^{i_1}(x)\otimes\dots\otimes\partial_0^{r}F^{i_s}(x), \tag{38}$$

each belonging to a partition $(r_1,\dots,r_s)$. Notice that to some partitions correspond summands that are originated from different partitions of $r-1$. For example, in degree 3 we have $(1,2)$-terms coming from the derivation of the terms of “type” 2 and others obtained from the derivation of the $(1,1)$-terms. We do not add summands of the same “type”, but keep them distinguished. By induction we assume that $\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$ is computed by the same algebraic expression as $\mathrm{d}^r H(j^r_{D_h}F(x))$, but writing in the summands in equation 38 $\partial^{r_j}_{\mathrm{sym}}\phi^{i_j}$ in place of $\partial_0^{r_j}F^{i_j}(x)$ and then evaluating at x.

To compute $\partial^{r+1}_{\mathrm{sym}}(H\circ\phi)(x)$ we have to firstly differentiate the algebraic expression that computes $\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$. From the previous assumption a one to one correspondence compatible with the partitions between the summands of $\mathrm{d}^{r+1}H(j^{r+1}_{D_h}F(x))$ and of $\nabla\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$ can be established. It is clear that restricting to D and taking the $(1,0)$ component does not affect to the identification.

In each summand of $\partial\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$ all the factors but possibly one in the tensor product are of the form $\partial^{r_j}_{\mathrm{sym}}\phi^{i_j}$ and hence already symmetric; the different one is of the form $\partial\partial^{r'_j}_{\mathrm{sym}}\phi^{i'_j}$.

Observe that the symmetrization of each summand in $\partial\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$ amounts to putting instead of $\partial\partial^{r'_j}_{\mathrm{sym}}\phi^{r'_j}$, its symmetrization $\partial^{r'_j+1}_{\mathrm{sym}}\phi^{r'_j}$ and then symmetrizing the resulting expression (this is an elementary result concerning symmetric products which is proved by suitably regrouping the permutations).

Thus we have proven that

$$\partial_{\mathrm{sym}}^{r+1}(H \circ \phi)(x) = \mathrm{sym}_{r+1}(\mathrm{d}^{r+1} H(j_D^{r+1}\phi(x))),$$

but $\mathrm{d}^{r+1}H(j_{D_h}^{r+1}F(x))$ is already symmetric. Therefore, we conclude $\partial_{\mathrm{sym}}^{r+1}(H \circ \phi)(x) = \mathrm{d}^{r+1}H(j_D^{r+1}\phi(x))$, where the equality also holds for each summand in the algebraic expression computing both quantities.

Hence the pseudo-holomorphic r-jet of a map to $\mathbb{CP}^m$ is well defined.

To be able to say when a sequence of functions of $\mathcal{J}_D^r(M, \mathbb{CP}^m)$ is A.H. we need to introduce an almost CR structure in the total space of the r-jets. This can be done using a connection (for example out of the Levi-Civita connection associated to the Fubini-Study metric in the projective space and of the connection on D^*). In our case we choose to do something different but equivalent: we use the identifications with $\mathcal{J}_D^r(M, \mathbb{C}^m)_i$. Each of these trivial vector bundles with trivial connection has a natural almost CR structure. Let $K_i \subset U_i$ be compact sets whose interiors cover $\mathbb{CP}^m$. We have the corresponding subsets $\mathcal{J}_D^r(M, \varphi_i^{-1}(K_i)) \subset \mathcal{J}_D^r(M, \mathbb{C}^m)_i$

We say that $\sigma_k \colon M \to \mathcal{J}_D^r(M, \mathbb{CP}^m)$ is A.H. if there exist constants $(C_j)_{j\geq 0}$ such that for all $x \in M$ and for all $j \geq 1$

$$\begin{array}{rcl} \max_{i\in\{0,\ldots,m\}} |\nabla^j (j^r\varphi_i^{-1} \circ \sigma_k)(x)|_{g_k} & \leq & C_j, \\ \max_{i\in\{0,\ldots,m\}} |\nabla^{j-1}\bar{\partial}(j^r\varphi_i^{-1} \circ \sigma_k)(x)|_{g_k} & \leq & C_j k^{-1/2}, \end{array}$$

where for each x we only take into account those indices for which $\sigma_k(x)$ belongs to the interior of $\mathcal{J}_D^r(M, K_i)$.

Notice that in the local models the identifications $j^r\Psi_{ji}$ are holomorphic, therefore when restricted to subsets associated to compact regions of $\mathbb{C}_i^m$ and $\mathbb{C}_j^m$ the sequence of maps $j^r\Psi_{ji} \colon \mathcal{J}_D^r(M, \mathbb{C}^m)_i \to \mathcal{J}_D^r(M, \mathbb{C}^m)_j$ is A.H. In particular, the notion of a sequence $\sigma_k \colon M \to \mathcal{J}_D^r(M, \mathbb{CP}^m)$ being A.H. does not depend on the covering K_i. It is also clear that if a sequence of functions ϕ_k is A.H., then $j_D^r\phi_k$ is also A.H.. This proves point 2 of the proposition.

If (P, Ω) is symplectic the definition of $\mathcal{J}^r(P, \mathbb{CP}^m)$ is the same (we just do not need to project the full derivative into the subspace D^*). When we have a J-complex distribution G there is an analogous definition of the bundle of pseudo-holomorphic r-jets along G. Using the previous affine coordinates of projective space we consider the sub-bundles

$$\mathcal{J}_G^r(P, \mathbb{C}^m)_i := (\sum_{j=0}^{r} (G^{*1,0})^{\odot j}) \otimes \underline{\mathbb{C}}^m,$$

where $\mathcal{J}_G^r(P, \mathbb{C}^m)_i \subset \mathcal{J}^r(P, \mathbb{C}^m)_i$ via the splitting $G \oplus G^\perp = TP$.

It is easily checked using the identification between $\mathcal{J}_{p,m}^r$ and $\mathcal{J}(P, \mathbb{C}^m)$ coming from approximately holomorphic coordinates adapted to G, that the diffeomorphisms $j^r\Psi_{ji} \colon \mathcal{J}^r(P, \mathbb{C}^m)_i \to \mathcal{J}^r(P, \mathbb{C}^m)_j$ preserve these sub-bundles.

The proof that shows that the $j^r\phi$ is well defined is exactly the same we gave for 2-calibrated manifolds; a small modification shows that $j_G^r\phi$ is well defined (instead of keeping the component ∇_D of the odd dimensional case, we project over G^*).

Going to the models furnished by approximately holomorphic coordinates adapted to G, the submersion $p_G \colon \mathcal{J}_{p,m}^r \to \mathcal{J}_{\mathbb{C}^g,p,m}^r$ is just a projection on some of the holomorphic coordinates, and therefore it is an approximately holomorphic sequence of maps.

Using approximately holomorphic coordinates adapted to G it is straightforward to check that if $\phi_k \colon P \to \mathbb{CP}^m$ is A.H., then both $j_G^r\phi_k$ and $j^r\phi_k$ are A.H. sequences of $\mathcal{J}^r(P, \mathbb{CP}^m)$. □

We recall that Z_k denotes the sequence of strata of $\mathcal{J}^r_D E_k$ (resp. $\mathcal{J}^r E_k$, $\mathcal{J}^r_G E_k$) of r-jets whose degree 0-component vanishes. We define $\mathcal{J}^r_D E_k^* := \mathcal{J}^r_D E_k \backslash Z_k$ (resp. $\mathcal{J}^r E_k^* := \mathcal{J}^r E_k \backslash Z_k$, $\mathcal{J}^r_G E_k^* := \mathcal{J}^r_G E_k \backslash Z_k$).

Proposition 6.

(1) *There exists a bundle map $j^r\pi\colon \mathcal{J}^r_D E_k^* \to \mathcal{J}^r_D(M, \mathbb{CP}^m)$ which is a fiberwise holomorphic submersion.*

(2) *For any section τ_k of E_k, in the points where it does not vanish and its projectivization ϕ_k is defined the following relation holds:*

$$j^r\pi(j^r_D\tau_k) = j^r_D\phi_k \tag{39}$$

In the almost complex case we have an analogous map $j^r\pi$, and for $\tau_k\colon P \to E_k$ the relation

$$j^r\pi(j^r\tau_k) = j^r(\pi \circ \tau_k) \tag{40}$$

Given G a J-complex distribution we have the following commutative square of submersions:

$$\begin{array}{ccc} \mathcal{J}^r E_k^* & \xrightarrow{p_G} & \mathcal{J}^r_G E_k^* \\ \downarrow{\scriptstyle j^r\pi} & & \downarrow{\scriptstyle j^r\pi} \\ \mathcal{J}^r(P, \mathbb{CP}^m) & \xrightarrow{p_G} & \mathcal{J}^r_G(P, \mathbb{CP}^m) \end{array} \tag{41}$$

If $j^r_G\tau_k$ is a section of $\mathcal{J}^r_G E_k^*$, for the restriction of $j^r\pi$ we have:

$$j^r\pi(j^r_G\tau_k) = j^r_G(\pi \circ \tau_k) \tag{42}$$

Proof. We define $j^r\pi$ to have the same expression as in the integrable case. That means that we fix approximately holomorphic coordinates and a section trivializing $L^{\otimes k}$ and a local frame of $E = \underline{\mathbb{C}}^{m+1}$, so that the r-jet σ in question is identified with the usual CR r-jet in a point x of a CR function F. Then $j^r\pi(\sigma)$ is defined to be the CR r-jet of $\pi \circ F$. Notice that for an appropriate chart φ_i^{-1} of projective space,

$$j^r\pi(\sigma) := \Pi^{-1}_{k,x,i}(j^r_{D_h}(\varphi_i^{-1} \circ \pi \circ F)(x)) \in \mathcal{J}^r_D(M, \mathbb{C}^m)_i \tag{43}$$

The arguments in proposition 5 that showed that the bundles $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ are well defined, also prove that $j^r\pi(\sigma)$ is well defined independently of the approximately holomorphic coordinates and of the chart of $\mathbb{CP}^m$ we used; it is as well independent of the local frame of E_k, because the map is equivariant w.r.t. the action of $Gl(m+1, \mathbb{C})$ on the fibers of E_k and on $\mathbb{CP}^m$.

It is clear that $j^r\pi$ is a submersion, and it is fiberwise holomorphic because in each fiber we have a map from some $\mathbb{C}^{m_1}$ to some $\mathbb{C}^{m_2}$ (after composing with a chart φ_i) whose formula is that of the integrable case which is trivially holomorphic, so point 1 holds.

We now prove the equality $j^r_D(\pi \circ \tau_k) = j^r\pi(j^r_D\tau_k)$: let φ_i^{-1} be any chart whose domain contains $\pi \circ \tau_k(x)$. Then by the definition given in proposition 5

$$j^r_D(\pi \circ \tau_k)(x) := j^r_D(\varphi_i^{-1} \circ \pi \circ \tau_k)(x)$$

We just defined in equation 43

$$j^r\pi(j^r_D\tau_k(x)) := \Pi^{-1}_{k,x,i}(j^r_{D_h}(\varphi_i^{-1} \circ \pi \circ F)(x))$$

By proposition 5 the r.h.s. of the two previous equalities coincide, i.e.

$$\Pi^{-1}_{k,x,i}(j^r_{D_h}(\varphi_i^{-1} \circ \pi \circ F)(x)) = j^r_D(\varphi_i^{-1} \circ \pi \circ \tau_k)(x)$$

Here the holomorphic function $\varphi_i^{-1} \circ \pi \colon \mathbb{C}^{m+1}\backslash\{0\} \to \mathbb{C}^m$ plays the role of H in proposition 5. Also observe that the proposition is in principle only valid when $\underline{\mathbb{C}}^{m+1}$ has the trivial connection. In the current situation $\underline{\mathbb{C}}^{m+1}$ is endowed with a diagonal connection coming from the one in $L^{\otimes k}$. The key point is that the composition $\varphi_i^{-1} \circ \pi \circ \phi_k$ is a section of $\underline{\mathbb{C}}^m \otimes L^{\otimes k} \otimes L^{-\otimes k}$ and hence a $\mathbb{C}^m$-valued function independently of the trivialization of $L^{\otimes k}$. Therefore the flat connection d on $\underline{\mathbb{C}}^m$ is induced from $\mathrm{d}\otimes\nabla_k$ in $\underline{\mathbb{C}}^{m+1}\otimes L^{\otimes k}$, where ∇_k is any hermitian connection on $L^{\otimes k}$. In other words, the equations of proposition 5 involving the connection $\nabla_g \otimes \mathrm{d}$ on $(T^{*1,0}\mathbb{C}^{n^{\odot r}}) \otimes \underline{\mathbb{C}}^{m+1}$ are also valid in this setting for the connection $\nabla_g \otimes (\mathrm{d} \otimes \nabla_k)$, and this finishes the proof of 2.

The previous ideas work word by word to show that for symplectic manifolds $j^r\pi\colon \mathcal{J}^r E_k^* \to \mathcal{J}^r(P, \mathbb{CP}^m)$ is a well defined submersion and that equation (40) holds.

If we have a distribution G, once we use the local identification coming from approximately holomorphic coordinates adapted to G, the commutativity of the diagram (41) follows from the commutativity in the holomorphic case. It is also clear that $j^r\pi\colon \mathcal{J}^r_G E_k^* \to \mathcal{J}^r_G(P, \mathbb{CP}^m)$ is a submersion and that equation 42 holds. $\square$

In order to describe the linearized Thom-Boardman stratification we need to define –at least for certain kinds of strata $\mathbb{P}S^a_k$ of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$– the corresponding subsets of transversal holonomy $\Theta_{\mathbb{P}S^a_k}$.

Definition 25. *Let $\mathbb{P}S_k$ be a sequence of strata of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ so that in canonical affine charts of $\mathbb{CP}^m$ and approximately holomorphic coordinates it is identified with a stratum $\mathbb{P}S$ of $\mathcal{J}^r_{D_h,n,m}$ invariant under the action of $\mathsf{T} \times Gl(n, \mathbb{C})$. Let $\mathbb{P}S_{k,i} =: \mathbb{P}S_k \cap \mathcal{J}^r_D(M, \mathbb{C}^m)_i$, $\Theta_{\mathbb{P}S_{k,i}} := \Theta_{\mathbb{P}S_k} \cap \mathcal{J}^r_D(M, \mathbb{C}^m)_i$. Then we define*

$$\Theta_{\mathbb{P}S_k} := \bigcup_{i\in\{0,\dots,m\}} \Theta_{\mathbb{P}S_{k,i}}$$

For the relative theory we assume that for a choice of approximately holomorphic coordinates adapted to G and canonical affine charts of projective space, the sequence $\mathbb{P}S_{k,i} \subset \mathcal{J}^r_G(P, \mathbb{C}^m_i)$ is identified with a stratum $\mathbb{P}S$ of $\mathcal{J}^r_{\mathbb{C}^g,p,m} = \mathcal{J}^r_{g,m} \times \mathbb{C}^{p-g}$ invariant under the action of $\mathsf{T}\times Gl(g, \mathbb{C})$. Then we define $\Theta_{\mathbb{P}S_k} = \bigcup_{i\in\{0,\dots,m\}} \Theta_{\mathbb{P}S_{k,i}}$.

Lemma 15. *The subsets $\Theta_{\mathbb{P}S_k}$ are well defined meaning that $j^r\Psi_{ji}(\Theta_{\mathbb{P}S_{k,i}}) \subset \Theta_{\mathbb{P}S_{k,j}}$, for each i,j.*

Therefore, if we define $S_k := j^r\pi^{-1}(\mathbb{P}S_k)$, with $j^r\pi\colon \mathcal{J}^r_D E^*_k \to \mathcal{J}^r_D(M, \mathbb{CP}^m)$ the submersion of proposition 6, the subsets $\check{\Theta}_{S_k} := j^r\pi^{-1}(\Theta_{\mathbb{P}S_k})$ are also well defined.

For jets along G also the subsets $\Theta_{\mathbb{P}S_k}$ are well defined. Let $S_k := j^r\pi^{-1}(\mathbb{P}S_k) \subset \mathcal{J}^r_G E^*_k$, $S^G_k := {p_G}^{-1}(S_k) \subset \mathcal{J}^r E^*_k$. We define the subset $\check{\Theta}_{S^G_k} \subset S^G_k$ by pulling back $\Theta_{\mathbb{P}S_k}$ to $\mathcal{J}^r E^*_k$ using either of the sides of the commutative diagram 41.

Proof. By lemma 14 the subsets $\Theta_{\mathbb{P}S_{k,i}}$ are well defined. Let us fix approximately holomorphic coordinates and canonical affine charts of $\mathbb{CP}^m$, so that $\Pi_{k,x,i}(\mathbb{P}S_{k,i}) = \mathbb{P}S$, for all k, x, i. We need to show that the subsets $\Theta_{\mathbb{P}S}$ glue well under the maps $j^r\Psi_{ji}$. Let ψ be an r-jet in $\Theta_{\mathbb{P}S}$. Then we have a lift $\tilde{\psi}$ to $\mathcal{J}^{r+1}_{D_h,n,m}$ and a local representation α of the lift cutting $\mathbb{P}S$ transversally along D_h at ψ. As we mentioned regarding transversality the local representation is essentially unique. That means in particular that any other representation α' will also share the transversality property. By definition $\tilde{\psi}$ is the (r+1)-jet of a local CR function F. Then $j^r_{D_h}F(0) = \psi$ and $(F(0), \mathrm{d}_{D_h} j^r_{D_h}F(0)) = (F(0), \partial_0 j^r_{D_h}F(0)) = j^{r+1}_{D_h}F(0) = \tilde{\psi}$. Thus, $j^r_{D_h}F$ is a local representation of $\tilde{\psi}$ which is transversal to $\mathbb{P}S$ along D_h at ψ.

Since $j^{r+1}\Psi_{ji}(j^{r+1}_{D_h}F) = j^{r+1}_{D_h}(\Psi_{ij}\circ F)$, $j^{r+1}\Psi_{ji}(\tilde{\psi})$ is a lift of $j^r\Psi_{ji}(\psi)$ with local representation $j^r_{D_h}(\Psi_{ij}\circ F)$, which is obviously transversal along D_h to $j^r\Psi_{ji}(\mathbb{P}S) =$

$\mathbb{P}S$ because $j^r\Psi_{ji}$ is a diffeomorphism that preserves the pullback of D_h to $\mathcal{J}^r_{D_h,n,m}$, and the proof of point 1 is completed.

A proof along the same lines shows the desired result for jets along G.

□

The Thom-Boardman-Auroux stratification is the pullback to $\mathcal{J}^r_D E^*_k$ by $j^r\pi$ of the analog of the Thom-Boardman stratification of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ (see for example [1, 7]), together with the strata Z_k. The definition is the natural extension of the one given for symplectic manifolds by D. Auroux in [4].

A first rough definition of the stratification of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ is the following: we fix approximately holomorphic coordinates and canonical affine charts of projective space, so we have charts $\Pi^{-1}_{k,x,i}\colon \mathcal{J}^r_{D_h,n,m} \to \mathcal{J}^r_D(M, \mathbb{C}^m)_i$.

In each $\mathcal{J}^r_{D_h,n,m}$ there is a CR Thom-Boardman stratification, which is $\mathsf{T} \times (\mathcal{H}^r_n \times \mathcal{H}^r_m)$-invariant, where $\mathcal{H}^r_l$ is the group of r-jets of germs of bi-holomorphic transformations from $\mathbb{C}^l$ to $\mathbb{C}^l$; in particular it is $\mathsf{T} \times Gl(n, \mathbb{C})$-invariant, so it defines an stratification on each $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$. The $\mathcal{H}^r_m$-invariance implies that the identifications that define $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ are compatible with the aforementioned stratifications on $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$.

Once we pullback the stratification to $\mathcal{J}^r_D E^*_k$ the behavior of the strata when they approach Z_k needs to be clarified. To do that we redefine the stratification as follows (see [4]):

Given $\sigma \in \mathcal{J}^r_D E^*_k$, let us denote its image in $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ by $\phi = (\phi_0, \ldots, \phi_r)$. Let us define

$$\Sigma_{k,i} = \{\sigma \in \mathcal{J}^r_D E^*_k | \dim_{\mathbb{C}} \ker \phi_1 = i\}$$

If $\max(0, n - m) < i \leq n$, the strata $\Sigma_{k,i}$ are smooth submanifolds whose boundary is the union $\bigcup_{j>i} \Sigma_{k,j}$ together with a subset of $Z_k \backslash \Theta_{Z_k}$.

Each $\Sigma_{k,i}$ is the pullback of a stratum $\mathbb{P}\Sigma_{k,i} \subset \mathcal{J}^r_D(M, \mathbb{CP}^m)$, and the given description of their closure is easy to check.

For $r \geq 2$, define $\check{\Theta}_{\Sigma_{k,i}}$ as the subset of r-jets $\sigma = (\sigma_0, \ldots, \sigma_r) \in \Sigma_{k,i}$ so that

$$\Xi_{k,i;\sigma} = \{u \in D, (i_u\phi, 0) \in T_\phi \mathbb{P}\Sigma_{k,i}\} \tag{44}$$

has the expected (complex) codimension in D, which is the (complex) codimension of $\Sigma_{k,i}$ in $\mathcal{J}^r_D E_k$ which equals the codimension of $\mathbb{P}\Sigma_{k,i}$ in $\mathcal{J}^r_D(M, \mathbb{CP}^m)$.

The subset $\check{\Theta}_{\Sigma_{k,i}}$ can be also defined using lemma 15: by definition $\Theta_{\mathbb{P}\Sigma_{k,i}}$ are exactly those points of $\mathbb{P}\Sigma_{k,i}$ which have a lift with a transversal local representation. Since the term that we add to the r-jet to define the lift is of order $r + 1 > 2$, the transversality of the local representation does not depend on the lift, that can be chosen to have vanishing component of order $r + 1$.

Fix as in the proof of lemma 14 A.H. coordinates so that at the origin $(D \oplus D^\perp, J) = (D_h \oplus D_v, J_0)$ and the induced connection form (on $\mathcal{J}^r_{D_h,n,m}$) is vanishing; fix also the canonical affine charts of $\mathbb{CP}^m$. Then the strata $\mathbb{P}\Sigma_{k,i}$ are sent to the Thom-Boardmman stratum $\mathbb{P}\Sigma_i$ of $\mathcal{J}^r_{D_h,n,m}$. The local representation of $(\phi, 0)$ can be taken to be α a CR section of $\mathcal{J}^r_{D_h,n,m}$. The stratum $\mathbb{P}\Sigma_i$ is CR, therefore

$$T_{D_h} j^1_{CR}\alpha(0) \cap (T\mathbb{P}\Sigma_i \cap \hat{D}_h)$$

is a complex subspace of $T\mathbb{C}^n$. Undoing the identifications, the previous subspace goes to the subspace in equation 44. By definition of transversality along D, $\Theta_{\mathbb{P}\Sigma_{k,i}}$ are exactly those ϕ for which $\Xi_{k,i;\sigma}$ has the codimension of $\mathbb{P}\Sigma_{k,i}$ in $\mathcal{J}^r_D(M, \mathbb{CP}^m)$. By construction (equation 44)

$$\check{\Theta}_{\Sigma_{k,i}} = j^r\pi^{-1}(\Theta_{\mathbb{P}\Sigma_{k,i}})$$

Hence $\check{\Theta}_{\Sigma_{k,i}}$ is the same subset introduced in lemma 15.

If $p+1 \leq r$, we define inductively

$$\Sigma_{k,i_1,\ldots,i_p,i_{p+1}} = \{\sigma \in \Theta_{\Sigma_{k,i_1,\ldots,i_p}}, \dim_{\mathbb{C}}(\ker\phi_1 \cap \Xi_{k,i_1,\ldots,i_p;\sigma}) = i_{p+1}\},$$

with

$$\Xi_{k,I;\sigma} = \{u \in D, (i_u\phi, 0) \in T_\phi \mathbb{P}\Sigma_{k,I}\}$$

As in the previous case, we define $\check{\Theta}_{\Sigma_{k,I}}$ either as the points such that the complex codimension of $\Xi_{k,I;\sigma}$ in D is the same as the codimension of $\Sigma_{k,I}$ in $\mathcal{J}^r_D E_k$, or as the pullback of $\Theta_{\mathbb{P}\Sigma_{k,I}}$.

If $i_1 \geq \cdots \geq i_{p+1} \geq 1$, $\Sigma_{k,i_1,\ldots,i_{p+1}}$ is –in the local model– a smooth CR submanifold whose closure in $\Sigma_{k,i_1,\ldots,i_p}$ is the union of the $\Sigma_{k,i_1,\ldots,i_p,j}, j > i_{p+1}$, and a subset of $\Sigma_{k,i_1,\ldots,i_p}\backslash\check{\Theta}_{\Sigma_{k,i_1,\ldots,i_p}}$ [7]. The problem is that for large values of r,n,m, the closure of the strata is hard to understand, and what we have defined –once Z_k has been added– might very well not be a Whitney quasi-stratification. For low values of r,n,m we have a Whitney quasi-stratification of $\mathcal{J}^r_D E_k$, because it comes from a Whitney stratification of $\mathcal{J}^r_D(M,\mathbb{CP}^m)$, and because the strata do not accumulate in points of Θ_{Z_k}.

Recall that using the local identifications the stratification we have defined (minus Z_k) is the union running over the affine charts of the pullback by $j^r(\varphi_i^{-1} \circ \pi)\colon \mathcal{J}^r_{D_h,n,m+1}\backslash Z \to \mathcal{J}^r_{D_h,n,m}$ of the CR Thom-Boardman stratification $\mathbb{P}\Sigma$ of $\mathcal{J}^r_{D_h,n,m}$. The latter is CR and $\mathsf{T}\times(Gl(n,\mathbb{C})\times\mathcal{H}^r_m)$-invariant

On the domain of each chart $\mathcal{J}^r_{D_h,n,m}$ we can use the results of Mather [25] to refine $\mathbb{P}\Sigma$ into a CR finite Whitney stratification transversal to the fibers and invariant under the action of $\mathsf{T}\times(Gl(n,\mathbb{C})\times\mathcal{H}^r_m)$, and such that the submanifolds $\mathbb{P}\Sigma_I$ are unions of strata of the refinement. Due to the required invariance properties for the refinements, they can be glued to give a refinement of the stratification $\mathbb{P}\Sigma_k \subset \mathcal{J}^r_D(M,\mathbb{CP}^m)$, which is independent of the choice of approximately holomorphic coordinates. Thus, its pullback is a finite Whitney stratification of $\mathcal{J}^r_D E^*_k$ and such that the $\Sigma_{k,I}$ are union of strata. It is by construction invariant by the action of $Gl(m+1,\mathbb{C})$ on the fiber.

It is important to notice that since all the strata are contained in the closure of $\Sigma_{k,\max(0,n-m)+1}$, they accumulate near Z_k in points of $Z_k\backslash\Theta_{Z_k}$. Therefore, by adding Z_k we obtain a quasi-stratification of $\mathcal{J}^r_D E_k$.

If we have a distribution G we use exactly the same definitions but in the subbundles $\mathcal{J}^r_G E_k$ and $\mathcal{J}^r_G(P,\mathbb{CP}^m)$. That is, we have the strata

$$\mathbb{P}\Sigma_{k,i} = \{\phi \in \mathcal{J}^r_G(P,\mathbb{CP}^m) | \dim_{\mathbb{C}}\ker\phi_1 = i\}$$

and for $r \geq 2$, $\check{\Theta}_{\mathbb{P}\Sigma_{k,i}} \subset \mathbb{P}\Sigma_{k,i}$ is the subset of r-jets along G $\phi = (\phi_0,\ldots,\phi_r)$ so that

$$\Xi_{k,i;\sigma} = \{u \in G, (i_u\phi, 0) \in T_\phi\mathbb{P}\Sigma_{k,i}\} \tag{45}$$

has the expected (complex) codimension in G, which is the (complex) codimension of $\mathbb{P}\Sigma_{k,i}$ in $\mathcal{J}^r_G(P,\mathbb{CP}^m)$.

The subsets $\mathbb{P}\Sigma_{k,I}$ are defined similarly. The result is a stratification $\mathbb{P}\Sigma_k$ of $\mathcal{J}^r_G(P,\mathbb{CP}^m)$. In charts adapted to G as in the proof of lemma 14 and affine charts -in which $\mathcal{J}^r_{\mathbb{C}^g,p,m} = \mathcal{J}^r_{g,m}\times\mathbb{C}^{p-g}$- the induced stratification $\mathbb{P}\Sigma$ is seen to be the leafwise Thom-Boardman stratification, i.e. the Thom-Boardman stratification of $\mathcal{J}^r_{g,m}$ multiplied by $\mathbb{C}^{p-g}$.

Using the lower part of the commutative diagram 41, we pullback back $\mathbb{P}\Sigma_k$ to $\mathbb{P}\Sigma^G_k \subset \mathcal{J}^r(P,\mathbb{CP}^m)$.

Let Σ^G_k be the pullback of $\mathbb{P}\Sigma^G_k$ to $\mathcal{J}^r E^*_k$. To refine it we first locally refine $\mathbb{P}\Sigma_k$ as follows: we go the leafwise Thom-Boardman stratification furnished by the previous A.H. coordinates and affine charts and construct a holomorphic $\mathsf{T}\times(Gl(g,\mathbb{C})\times\mathcal{H}^r_m)$-invariant refinement in one of the leaves of $\mathcal{J}^r_{\mathbb{C}^g,p,m}$ (which is identified with

$\mathcal{J}^r_{g,m}$). Next we extend it independently of the remaining $p-g$ complex coordinates $z_k^{g+1},\ldots,z_k^p$. The local refinements of the leafwise Thom-Boardman stratification glue well and thus define a sequence of Whitney stratifications $\mathcal{J}^r_G(P,\mathbb{CP}^m)$ which does not depend either on the A.H. coordinates adapted to G or in the chosen affine charts of $\mathbb{CP}^m$. Its pullback to $\mathcal{J}^r E_k^*$ refines Σ_k^G to a sequence of Whitney stratifications.

Definition 26. *(see [4]).*

(1) *Given (M,D,J,g_k) and $E_k=\underline{\mathbb{C}}^{m+1}\otimes L^{\otimes k}$, the Thom-Boardman-Auroux stratification of $\mathcal{J}^r_D(M,\mathbb{CP}^m)$, denoted by $\mathbb{P}\Sigma_k$, is the stratification (or rather its refinement) built out of the pieces of the Thom-Boardman stratifications of $\mathcal{J}^r_{D_h,n,m}$. The Thom-Boardman-Auroux quasi-stratification of $\mathcal{J}^r_D E_k$ is the pullback of the Thom-Boardman-Auroux stratification of $\mathcal{J}^r_D(M,\mathbb{CP}^m)$ together with the zero section. We denote it by Σ_k.*

(2) *Let (P,J,G,g_k) and $E_k=\underline{\mathbb{C}}^{m+1}\otimes L^{\otimes k}$. The Thom-Boardman-Auroux stratification of $\mathcal{J}^r(M,\mathbb{CP}^m)$ along G, denoted by $\mathbb{P}\Sigma_k^G$, is the stratification (or rather its refinement) built out of the pieces of the Thom-Boardman stratifications of $\mathcal{J}^r_{\mathbb{C}^g,p,m}$. The Thom-Boardman-Auroux quasi-stratification of $\mathcal{J}^r E_k$ along G, that we denote by Σ_k^G, is the pullback of the Thom-Boardman-Auroux stratification of $\mathcal{J}^r(M,\mathbb{CP}^m)$ along G together with Z_k.*

Lemma 16. *The Thom-Boardman-Auroux quasi-stratification of $\mathcal{J}^r_D E_k$ and the Thom-Boardman-Auroux quasi-stratification of $\mathcal{J}^r E_k$ along G are approximately holomorphic (and also finite and Whitney).*

Proof. We start with jets along D. The description of the closure of the strata inside Z_k implies that the quasi-stratification condition holds.

The delicate point is checking that the strata are approximately holomorphic (for the modified connection).

First we study the sequence Z_k. Though for this sequence the approximate holomorphicity is obvious, we will give a proof that works for other sequences of strata.

We first prove that the natural projection $\pi^r_{r-h}\colon \mathcal{J}^r_D E_k\to\mathcal{J}^{r-h}_D E_k$ is approximately holomorphic: we fix A.H. coordinates and A.H. reference frames $j^r_D\tau^{\mathrm{ref}}_{k,x,I}$ of $\mathcal{J}^r_D E_k$ (resp. $j^r_D\tau^{\mathrm{ref}}_{k,x,I'}$ of $\mathcal{J}^{r-h}_D E_k$) as in equation 25. Recall that proposition 4 implies that the sequences are indeed A.H. Using these frames we obtain A.H. coordinates $z_k^1,\ldots,z_k^n,u_k^I,s_k$ (resp. $z_k^1,\ldots,z_k^n,v_k^{I'},s_k$) for the total space of $\mathcal{J}^r_D E_k$ (resp. $\mathcal{J}^{r-h}_D E_k$). From the equality $\pi^r_{r-h}(j^r_D\tau^{\mathrm{ref}}_{k,x,I})=j^{r-h}_D\tau^{\mathrm{ref}}_{k,x,I}$ we deduce $\pi^r_{r-h}(j^r_D\tau^{\mathrm{ref}}_{k,x,I})=W_I(z_k,v_k^{I'})$, where $W_I(z_k,v_k^{I'})$ is A.H. w.r.t the canonical CR structures associated to the coordinates. This, together with the fiberwise linearity of π^r_{r-h} imply that in these coordinates π^r_{r-h} is A.H., and hence it is A.H. w.r.t. the almost CR structures of the total spaces.

We want to do something similar with the strata $\Sigma_{k,I}$ and the projection $j^r\pi\colon\mathcal{J}^r_D E_k^*\to\mathcal{J}^r_D(M,\mathbb{CP}^m)$ (away from a uniform tubular neighborhood of the zero section, where the differential goes to infinity). The image of a trivialization $j^r_D\tau^{\mathrm{ref}}_{k,x,I}$ is $j^r_D(\pi\circ\tau^{\mathrm{ref}}_{k,x,I})$, also approximately holomorphic. The map is equally fiberwise holomorphic, but the difference is the non-linearly of the restriction to the fibers.

We adopt a different strategy that amounts to perturbing the almost CR structures into integrable ones and then checking that $j^r\pi$ is CR w.r.t. them: we take Darboux charts and trivialize $L^{\otimes k}$ with a unitary section ξ_k whose associated connection form in the domain of Darboux charts is A. Next we trivialize $\mathcal{J}^r_D E_k$ with the frames $\mu_{k,x,I}$ of equation 24, but using ξ_k tensored with a basis of $\mathbb{C}^{m+1}$ to

trivialize $\underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$. In this way $\mathcal{J}^r_D E_k$ becomes the trivial bundle $\mathcal{J}^r_{D_h,n,m+1}$ (with is canonical trivialization constructed out of $dz_k^1, \ldots, dz_k^n$). Let us use in the base the canonical CR structure (D_h, J_0). Proposition 4 in the integrable case (and for curvature of type $(1,1)$ and with trivial derivative, as it is the case in Darboux coordinates) implies that the modified connection defines a new CR structure in the total space of $\mathcal{J}^r_{D_h,n,m}$; let $(\hat{D}_h, \bar{J}_0)$ be the corresponding distribution and almost complex structure, and let $(\hat{D}, \hat{J})$ be the distribution and almost complex structure induced by the almost CR structure of $\mathcal{J}^r_D E_k$. If in the fiber of $\mathcal{J}^r_{D_h,n,m+1}$ we fix a ball $B(\sigma, R)$, then in $B(0, O(1)) \times B(\sigma, R)$ the Euclidean metric is comparable with the metric carried by $\mathcal{J}^r_D E_k$. More important

$$|\mathrm{d}^j(\hat{D} - \hat{D}_h)|_{g_0} \leq O(k^{-1/2}),\ j \geq 0 \tag{46}$$

And if we use the orthogonal projection to push $\hat{J}$ into $\hat{J}_h \colon \hat{D}_h \to \hat{D}_h$, we also have

$$|\mathrm{d}^j(\hat{J}_h - \bar{J}_0)|_{g_0} \leq O(k^{-1/2}),\ j \geq 0 \tag{47}$$

We use the same Darboux charts for $\mathcal{J}^r_{D_h}(\mathbb{C}^n \times \mathbb{R}, \mathbb{CP}^m)$, so locally and using canonical affine charts we have identifications with $\mathcal{J}^r_{D_h,n,m}$. This is a trivial vector bundle (again using the basis induced $dz_k^1, \ldots, dz_k^n$ and the basis of $\mathbb{C}^m$). We fix the product CR structure and denote by $(\tilde{D}_h, \tilde{J}_0)$ the distribution and almost complex structure. Let $(\tilde{D}, \tilde{J})$ be the distribution and almost complex structure induced by the almost CR structure of $\mathcal{J}^r_D(M, \mathbb{CP}^m)$. By construction

$$|\mathrm{d}^j(\tilde{D} - \tilde{D}_h)|_{g_0}, |\mathrm{d}^j(\tilde{J}_h - \tilde{J}_0)|_{g_0} \leq O(k^{-1/2}),\ j \geq 0, \tag{48}$$

where $\tilde{J}_h$ is the almost complex structure on $\tilde{D}_h$ defined out of $\tilde{J}$ and the orthogonal projection.

Equations 46, 47, 48 imply that if $j^r(\varphi_i^{-1} \circ \pi) \colon \mathcal{J}^r_{D_h,n,m+1} \to \mathcal{J}^r_{D_h,n,m}$ is CR w.r.t. $(\hat{D}_h, \bar{J}_0)$ and $(\tilde{D}_h, \tilde{J}_0)$ then it is almost CR w.r.t the global almost CR structures.

The map $j^r(\varphi_i^{-1} \circ \pi) \colon \mathcal{J}^r_{D_h,n,m+1} \to \mathcal{J}^r_{D_h,n,m}$ is exactly the same as in the holomorphic (or rather CR) models. It is CR w.r.t. the aforementioned CR structures because it preserves the foliations, it is fiberwise holomorphic and sends "enough" CR sections of $\mathcal{J}^r_{D_h,n,m+1}$ to CR sections of $\mathcal{J}^r_{D_h,n,m}$. To be more precise, for any point $\sigma \in \mathcal{J}^r_{D_h,n,m+1}$ and any vector v in its tangent space along the leaf and not tangent to the fiber, we can find a CR section F whose CR r-jet in x is σ and such that the tangent space to its graph contains v. Since $j^r(\varphi_i^{-1} \circ \pi)(j^r_{D_h} F) = j^r_{D_h}(\varphi_i^{-1} \circ \pi \circ F)$ is also a CR section, we deduce that $j^r(\varphi_i^{-1} \circ \pi)_*(\bar{J}v) = \tilde{J}_0(j^r(\varphi_i^{-1} \circ \pi)_*(v))$.

The strata $\mathbb{P}\Sigma_k$ (or rather of its refinement)-once we choose A.H. coordinates and affine charts of projective space- are identified with the strata of (the refinement of) the CR Thom-Boardman stratification of $\mathcal{J}^r_{D_h,n,m}$, with are CR. The comparison between the $(\hat{D}_h, \bar{J}_0, g_0)$ and the original almost CR structure imply that the strata of $\mathbb{P}\Sigma_k$ are A.H., and hence $\Sigma_k = j^r\pi^{-1}(\mathbb{P}\Sigma_k)$ is A.H. More precisely, if f is a local CR function defining a CR strata $\mathbb{P}\Sigma_I$ of $\mathcal{J}^r_{D_h,n,m}$, then $f \circ \Pi_{k,x,i} \circ j^r(\varphi_i^{-1} \circ \pi)$ is a sequence of local functions for the strata $\Sigma_{k,I}$. Therefore condition 3 in definition 18 holds. The other ones follow from the fact that we have a model Whitney stratification transversal to the fibers (see the proof of lemma 13).

There is an analogous proof in the almost complex setting showing that $j^r\pi \colon \mathcal{J}^r E_k^* \to \mathcal{J}^r(P, \mathbb{CP}^m)$ is approximately holomorphic away from a uniform neighborhood of the zero section. In the relative case, and for a sequence of A.H. strata $\mathbb{P}S_k$ fulfilling the conditions of definition 25, the approximate holomorphicity of ${p_G}^{-1} j^r \pi^{-1} S_k$ follows from the commutativity of the diagram 41, and from the approximate holomorphicity of $j^r\pi \colon \mathcal{J}^r E_k^* \to \mathcal{J}^r(P, \mathbb{CP}^m)$ and of $p_G \colon \mathcal{J}^r(P, \mathbb{CP}^m) \to \mathcal{J}^r_G(P, \mathbb{CP}^m)$. Recall that the strata $\mathbb{P}\Sigma_k$ come from holomorphic models (the refinement of the

strata of the leafwise Thom-Boardman stratification), so they are A.H. But Σ_k^G is not truly a quasi-stratification of $\mathcal{J}^r E_k$. To be more precise, it is not true that the strata only accumulate in points of $Z_k \backslash \Theta_{Z_k} \subset Z_k$, but it is still true that the points of Z_k in which the other strata accumulate are never hit by a section transversal to Z_k along G. Thus, the Whitney type reasoning can be applied as long as we work with r-jets along G (see the proof of theorem 3). □

Remark 13. *Notice that we only conclude that the strata different form the zero section are approximately holomorphic uniformly far from Z_k. This is enough for our purposes, for once we obtain transversality to Z_k our r-jet will be uniformly far from $Z_k \backslash \Theta_{Z_k}$. All the remaining strata approach Z_k accumulating only on points of $Z_k \backslash \Theta_{Z_k}$. Therefore, the r-jet will only hit them outside of a uniform tubular neighborhood of Z_k, where the approximate holomorphicity holds.*

Definition 27.

(1) *An A.H. sequence of sections of $E_k \to (M, D, J, g_k)$ is said to be r-generic if its pseudo-holomorphic r-jet is uniformly transversal along D to the Thom-Boardman-Auroux quasi-stratification of $\mathcal{J}^r_D E_k$.*

(2) *An A.H. sequence of sections of $E_k \to (P, J, G, g_k)$ is said to be r-G-generic along M if its pseudo-holomorphic r-jet is uniformly transversal along M to $\Sigma_k^G \subset \mathcal{J}^r E_k$.*

(3) *Let $\phi_k \colon M \backslash B_k \to \mathbb{CP}^m$ be sequence of functions which is A.H. outside of a uniform tubular neighborhood of g_k-radius $O(1)$ of B_k. It is said to be r-generic if for k large enough B_k is a codimension $2(m+1)$ calibrated submanifold and $j^r_D \phi_k \colon M \backslash B_k \to \mathcal{J}^r_D(M \backslash B_k, \mathbb{CP}^m)$ is uniformly transversal along D to the Thom-Boardman-Auroux stratification. Moreover, it is required to intersect the strata of strictly positive codimension out of the previous tubular neighborhood of B_k of g_k-radius $O(1)$.*

Lemma 17. *Let τ_k be an A.H. sequence of sections of $E_k \to (M, D, J, g_k)$. Then if τ_k is r-generic its projectivization $\phi_k \colon M \backslash \tau_k^{-1}(Z_k) \to \mathbb{CP}^m$ is also r-generic.*

Proof. It is elementary from the construction of the Thom-Boardman-Auroux (quasi)-stratifications of $\mathcal{J}^r_D E_k$ and $\mathcal{J}^r_D(M, \mathbb{CP}^m)$, proposition 6 relating $j^r_D \tau_k$ and $j^r_D \phi_k$ and lemma 16.

Uniform transversality of τ_k to Z_k implies by remark 13 that ϕ_k intersects the remaining strata uniformly away from the zero set. Estimated transversality along D is also preserved when composed with $j^r \pi$ uniformly away from Z; the key point is selecting appropriate local A.H. functions for the strata: in A.H. coordinates and affine charts $\mathbb{P}\Sigma_{k,I}$ corresponds to a CR stratum $\mathbb{P}\Sigma_I$. Let f be a local CR function defining it. Then $f \circ \Pi_{k,x,i} \circ j^r(\varphi_i^{-1} \circ \pi)$ are local functions for $\Sigma_{k,I}$. Now lemma 11 implies that local uniform estimated transverslity of $j^r_D \tau_k$ to $\Sigma_{k,I}$ along D is equivalent to uniform transversality to $\mathbf{0}$ along D of $f \circ j^r(\varphi_i^{-1} \circ \pi) \circ j^r_D \tau_k = f \circ j^r_D(\varphi_i^{-1} \circ \phi_k)$. Again by the same lemma this is equivalent to uniform transversality of $j^r_D \phi_k$ along D to $\mathbb{P}\Sigma_{k,I}$. The case of the points close to the boundary of the strata is just a problem in a vector space; it follows from $j^r(\varphi_i^{-1} \circ \pi) \colon \mathcal{J}^r_{D_h,n,m+1} \backslash Z \to \mathcal{J}^r_{D_h,n,m}$ being a submersion which amounts to suppress coordinates of the fiber of $\mathcal{J}^r_{D_h,n,m+1}$.

□

Let (P, Ω) be a symplectic manifold with $(M, D, \omega := \Omega_{|M})$ 2-calibrated and G a local J-complex distribution extending D.

Let τ_k be an A.H. sequence of sections of E_k and denote by ϕ_k its projectivization away from its zero set.

Proposition 7. *Using the above notation, if $j^r\tau_k\colon P \to \mathcal{J}^r E_k$ is uniformly transversal along M to $\Sigma_k^G \subset \mathcal{J}_G^r E_k$, then $\phi_{k|M}$ is r-generic.*

Proof. We will make extensive use of diagram 41

$$\begin{array}{ccc} \mathcal{J}^r E_k^* & \xrightarrow{p_G} & \mathcal{J}_G^r E_k^* \\ \big\downarrow j^r\pi & & \big\downarrow j^r\pi \\ \mathcal{J}^r(P,\mathbb{CP}^m) & \xrightarrow{p_G} & \mathcal{J}_G^r(P,\mathbb{CP}^m) \end{array}$$

Step 1: Study the compatibility of the Thom-Boardman-Auroux stratifications with the identification of $\mathcal{J}_D^r(M,\mathbb{CP}^m)$ with $\mathcal{J}_G^r(P,\mathbb{CP}^m)_{|M}$.

In the points of M there is a canonical J-complex identification between D and G, inducing *isometries*

$$\Lambda_{k,i}\colon \mathcal{J}_D^r(M,\mathbb{CP}^m) \to \mathcal{J}_G^r(P,\mathbb{CP}^m)_{|M}$$

Let $z_k^1,\dots,z_k^p$ be A.H coordinates adapted to M. We can rewrite them as $z_k^1,\dots,z_k^n,x_k^{2n+1},x_k^{2n+2},z_k^{n+2},\dots,z_k^p$, where $z_k^1,\dots,z_k^n,x_k^{2n+1}$ are by lemma 6 A.H. coordinates for M. Using also the canonical affine charts of projective space we have

$$\begin{array}{rcl} \Pi_{k,x,i}^D\colon \mathcal{J}_D^r(M,\mathbb{C}^m)_i & \to & \mathcal{J}_{D_h,n,m}^r = \mathcal{J}_{n,m}^r \times \mathbb{R} \\ \Pi_{k,x,i}^G\colon \mathcal{J}_G^r(P,\mathbb{C}^m)_i & \to & \mathcal{J}_{\mathbb{C}^n,p,m}^r = \mathcal{J}_{n,m}^r \times \mathbb{C}^{p-n}, \end{array}$$

and a canonical identification in $\mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^p$

$$\Lambda\colon \mathcal{J}_{D_h,n,m}^r \to \mathcal{J}_{\mathbb{C}^n,p,m|\mathbb{C}^n\times\mathbb{R}}^r$$

The construction of $\Pi_{k,x,i}^D\Pi_{k,x,i}^G$ (see equation 21 and the last paragraph in the proof of lemma 14) implies the commutativity of

$$\begin{array}{ccc} \mathcal{J}_D^r(M,\mathbb{CP}^m) & \xrightarrow{\Lambda_k} & \mathcal{J}_G^r(P,\mathbb{CP}^m)_{|M} \\ \big\downarrow \Pi_{k,x,i}^D & & \big\downarrow \Pi_{k,x,i}^G \\ \mathcal{J}_{D_h,n,m}^r & \xrightarrow{\Lambda} & \mathcal{J}_{\mathbb{C}^n,p,m|\mathbb{C}^n\times\mathbb{R}}^r \end{array} \tag{49}$$

And the restriction of $\mathcal{J}_{\mathbb{C}^n,p,m}^r$ to $\mathbb{C}^n\times\mathbb{R} \approx M$ coincides with $\mathcal{J}_{n,m}^r\times\mathbb{R} = \mathcal{J}_{D_h,n,m}^r$.

The identification Λ obviously preserves the Thom-Boardman-Auroux stratifications (and even the refinements), and hence so Λ_k does.

Step 2: Check that $\Lambda_k^{-1}\circ(j_G^r\phi_k)_{|M} \approxeq j_D^r(\phi_{k|M})$.

Since Λ_k are isometries preserving the Thom-Boardman-Auroux stratifications, we omit them from now on.

By using the charts $\Pi_{k,x,i}^D\Pi_{k,x,i}^G$ it is easy to see that for any $j \in \{1,\dots,r\}$, the degree j homogeneous component of $j_D^r(\phi_{k|M})$ approximately coincides with $\nabla_D^j(\phi_{k|M})$. Similarly, the degree j homogeneous component of $j_G^r\phi_k$ approximately coincides with $\nabla_G^j\phi_k$. The result follows because we also have

$$(\nabla_G^j\phi_k)_{|M} \approxeq \nabla_D^j(\phi_{k|M})$$

Step 3: Analyze the behavior of $j_D^r(\phi_{k|M})$ near the set of base points B_k.

Since $Z_k \subset \mathcal{J}^r E_k$ is an A.H. sequence of submanifolds and $j^r\tau_k$ an A.H. sequence of sections, by corollary 6 uniform transversality along M is equivalent to uniform transversality along G in the points of M. In A.H. coordinates adapted to G, we are

saying that the matrix of partial derivatives of τ_k w.r.t. $z_k^1, \dots, z_k^g$ has maximum rank and norm greater than some $\eta > 0$. But this is equivalent to saying that is uniformly transversal to Z_k^G, the pullback of the zero section of $\mathcal{J}_G^r E_k$.

By construction $\Sigma_k^G \backslash Z_k = p_G^{-1} j^r \pi^{-1}(\mathbb{P}\Sigma_k) = p_G^{-1}(\Sigma_k \backslash Z_k)$, and the strata of $\Sigma_k^G \backslash Z_k$ when approaching the zero section accumulate into $p_G^{-1}(\Theta_{Z_k})$, where here $\Theta_{Z_k} \subset \mathcal{J}_G^r E_k$. Therefore $j^r \tau_k$ intersects the strata of $\Sigma_k^G \backslash Z_k$ away from a tubular neighborhood in P (and hence in M) of radius $O(1)$ of B_k, the zero set of $j^r \tau_k$. Thus $(j^r \phi_k)_{|M} = (j^r \pi(j^r \tau_k))_{|M}$ intersects the strata of $\mathbb{P}\Sigma_k^G$ away from a tubular neighborhood in M of radius $O(1)$ of B_k.

In general $p_G(j^r \phi_k) \neq j_G^r \phi_k$ but using A.H. coordinates it is easy to check that $p_G(j^r \phi_k) \approxeq j_G^r \phi_k$. Hence, $j_G^r \phi_k$ intersects the strata of $\mathbb{P}\Sigma_k \subset \mathcal{J}_G^r(P, \mathbb{CP}^m)$ away from a tubular neighborhood in M of radius $O(1)$ of B_k, for all $k \gg 1$.

By steps 1 and 2 we deduce that $j_D^r(\phi_{k|M})$ intersects the strata of $\mathbb{P}\Sigma_k \subset \mathcal{J}_D^r(P, \mathbb{CP}^m)$ away from a tubular neighborhood in M of radius $O(1)$ of B_k, for all $k \gg 1$.

Step 4: Relate uniform transversality of $j^r \tau_k$ along M to $\Sigma_k^G \backslash Z_k$ with uniform transversality of $j_D^r(\phi_{k|M})$ along D to $\mathbb{P}\Sigma_k \subset \mathcal{J}_D^r(M, \mathbb{CP}^m)$.

The same ideas used in the proof of lemma 17 combined $p_G(j^r \phi_k) \approxeq j_G^r \phi_k$, show that uniform transversality of $j^r \tau_k$ along M to $\Sigma_k^G \backslash Z_k$ is equivalent to uniform transversality along M of $j_G^r \phi_k$ to $\mathbb{P}\Sigma_k \subset \mathcal{J}_G^r(P, \mathbb{CP}^m)$.

Uniform transversality along M of $j_G^r \phi_k$ to $\mathbb{P}\Sigma_k \subset \mathcal{J}_G^r(P, \mathbb{CP}^m)$ is comparable to uniform transversality of $(j_G^r \phi_k)_{|M}$ to $\mathbb{P}\Sigma_{k|M} \subset \mathcal{J}_G^r(P, \mathbb{CP}^m)_{|M}$ (it can be easily proven in the charts $\Pi_{k,x,i}^D \Pi_{k,x,i}^G$).

Steps 1 and 2 imply that $j_D^r(\phi_{k|M})$ is uniformly transversal to $\mathbb{P}\Sigma_k \subset \mathcal{J}_D^r(M, \mathbb{CP}^m)$.

If the hypothesis on the amount of transversality along M of corollary 6 are met, then $j_D^r(\phi_{k|M})$ is uniformly transversal along D to $\mathbb{P}\Sigma_k \subset \mathcal{J}_D^r(M, \mathbb{CP}^m)$. Observe that this requirement is not a problem, since the induction construction to obtain uniform transversality along M for $j^r \tau_k$ to $\Sigma_k^G \backslash Z_k$ can guarantee that. $\square$

The (modified) connection and its associated metric the total space of $\mathcal{J}^r E_k$ induce on $\mathcal{J}_G^r E_k$ via p_G a metric (with restricts fiberwise to the hermitian bundle metric) and a connection; we denote them by $\hat{g_k}$ and $\nabla_{k,H}$ (or just ∇_H); we do not know whether $\mathcal{J}_G^r E_k$ is an almost CR submanifold of $\mathcal{J}^r E_k$, but in any case we are not interested in fixing any almost complex structure on $\mathcal{J}_G^r E_k$.

Let σ_k be a sequence of sections of $\mathcal{J}_G^r E_k$ with $|\nabla^j \sigma_k|_{g_k} \leq O(1)$, $\forall j \geq 0$. Using the metric $\hat{g_k}$ we have a well defined notion of uniform transversality of σ_k to the Thom-Boardman-Auroux stratification $\Sigma_k \subset \mathcal{J}_G^r E_k$ (definition 17); notice that we have no notion of approximate holomorphicity neither for the sequence of sections nor for the strata.

Remark 14. *If $\tau_k \colon P \to E_k$ is A.H. then $|\nabla^j j_G^r \tau_k|_{g_k} \leq O(1)$, $\forall j \geq 0$. Having into account remark 8, it can also be shown that if $j^r \tau_k \colon P \to \mathcal{J}^r E_k$ is uniformly transversal along M to Σ_k^G, then $j_G^r \tau_k \colon P \to \mathcal{J}_G^r E_k$ is uniformly transversal along M to Σ_k.*

We finish this section by proving the following

Lemma 18.

(1) *Let $\mathcal{S} = S_k^a$ be an approximately holomorphic finite invariant stratification of E_k such that in approximately holomorphic coordinates and A.H. frames each sequence of strata has a CR model transversal to the fibers. Let $\tau_k \colon M \to E_k$ be an A.H. sequence uniformly transversal along D to $\mathcal{S}$. Then $\tau_k^{-1}(\mathcal{S})$ is an stratification of (M, D, ω) by 2-calibrated submanifolds for all $k \gg 1$.*

(2) *Let $\tau_k\colon M \to E_k$ be an A.H. uniformly transversal to Z_k and whose projectivization ϕ_k is r-generic. Then $B_k \cup \phi_k^{-1}(\mathbb{P}\Sigma_k)$ is a stratification by 2-calibrated submanifolds of (M, D, ω) for all $k \gg 1$.*

Proof. Let $S_k^a \subset E_k$. Corollary 5 implies that $\tau_k^{-1}(S_k^a)$ is uniformly transversal to D. Hence, if we check that for each $x \in \tau_k^{-1}(S_k^a)$ the sequence of linear subspaces $T_D\tau_k^{-1}(S_k^a) \subset D$ is A.H., i.e. $\angle_M(T_D\tau_k^{-1}(S_k^a), JT_D\tau_k^{-1}(S_k^a)) \leq O(k^{-1/2})$ (uniformly on the point), we are done.

Let $\hat{J}$ denote the induced the almost complex structure on E_k. In approximately holomorphic coordinates and A.H. frames the strata $S_k \subset E_k$ have a CR model $S \subset \underline{\mathbb{C}}^m$ w.r.t the canonical product CR structure. Recall that any almost CR structure defined out of J_0 in the base and the fiber and a connection form with vanishing $(0,1)$-component coincides with the product CR structure. (this appears also in the proof of lemma 13). Hence, the linear subspaces $T_DS = T_DS_k$ verify $\angle_M(T_DS, \hat{J}T_DS) \leq O(k^{-1/2})$, *the bounds being uniform on the points of* $\underline{\mathbb{C}}^m$, and hence uniform on the points of E_k.

The approximate holomorphicity of τ_k implies $\angle_M(T_D\tau_k, \hat{J}T_D\tau_k) \leq O(k^{-1/2})$.

Since $\angle_m(T_D\tau_k, T_DS_k) \geq \eta$, by proposition 3.7 in [28] for all $k \gg 1$ the intersection $T_D\tau_k \cap T_DS_k$ is an A.H. sequence and thus also its projection to M, and this proves point 1.

Regarding point 2, $B_k := {\tau_k}^{-1}(Z_k)$. Therefore point 1 applies.

The strata $\Sigma_{k,I}$ are intersected uniformly away from B_k. Hence, it is equivalent to work with the projectivizations ϕ_k and the Thom-Boardman-Auroux stratification of $\mathcal{J}_D^r(M, \mathbb{CP}^m)$, because $j_D^r\tau_k^{-1}(\Sigma_{k,I}) = j_D^r\phi_k^{-1}(\mathbb{P}\Sigma_{k,I})$. Since for each canonical chart of projective space the strata have CR models in $\mathcal{J}_D^r(M, \mathbb{C}^m)_i$, everything reduces to point 1. □

We would like the pullback of any regular value of ϕ_k to be a 2-calibrated submanifold, which forces us to study the behavior of an r-generic function near its base locus and near the pullback of the Thom-Boardman-Auroux strata. Our applications only have to do with 1-jets, and only the Lefschetz pencils $\phi_k\colon M\backslash B_k \to \mathbb{CP}^1$ would have non-empty base locus. The same ideas used in [32] show that indeed near the base locus $|\partial\phi_k| > |\bar{\partial}\phi_k|$ and thus the regular "fibers" are 2-calibrated submanifolds. On the other hand, near the strata of the Thom-Boardman-Auroux stratification there is no such inequality between the holomorphic and antiholomorphic component of the derivative, and ad hoc modifications are needed to obtain 2-calibrated regular fibers.

Observe that if the approximately holomorphic theory is used to construct generic CR sections for of a Levi-flat CR manifold, this kind of complication near the base locus and degeneration loci of the leafwise differential does not occur.

7. The main theorem

It is possible to perturb A.H. sections of $E_k = E \otimes L^{\otimes k} \to (M, D, \omega)$ so that their r-jets are transversal to an A.H. quasi-stratification of $\mathcal{J}_D^r E_k$.

Theorem 2. *Let $E_k \to (M, D, \omega)$, $E_k = E \otimes L^{\otimes k}$ and $\mathcal{S} = (S_k^a)_{a \in A_k}$ an A.H. sequence of finite Whitney quasi-stratifications of $\mathcal{J}_D^r E_k$ transversal to the fibers. Let us fix $h \in \mathbb{N}$. Let δ be an strictly positive constant. Then a constant $\eta > 0$ exists such that for any A.H. sequence τ_k of E_k, it is possible to find an A.H. sequence σ_k of E_k so that for every k bigger than some k_0,*

(1) $|\nabla_D^j(\tau_k - \sigma_k)|_{g_k} < \delta, j = 0, \ldots, r+h$
(2) $j_D^r\sigma_k$ *is η-transversal along D to $\mathcal{S}$*

Theorem 3 -to be introduced- suffices for our applications; the proof of theorem 2 -which is left to the interested reader- is a suitable modification of the proof of theorem 1.1. in [4], being the main difference the use of a local estimated transversality along D_h to $\mathbf{0}$ for A.H. functions $f_k\colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^m$.

Observe that while for any $h \in \mathbb{N}$ we can bound $|\nabla^j_D(\tau_k - \sigma_k)|_{g_k}$, $j = 0, \ldots, r+h$ by any arbitrarily small δ, we cannot do the same for the full derivative. For that we just know $|\nabla^j_D(\tau_k - \sigma_k)|_{g_k} \leq C_j$, $\forall j \in \mathbb{N}$, with no control on the constants C_j. Moreover, the non-integrability of D also forces us to work with sequences of A.H. functions all whose derivatives are controlled (even if we want to control the size of the perturbation along D up to a finite order h); basically the derivatives along the directions of D (up to some finite order h) will be arbitrarily small only if we have control for the full derivative of all the orders, and k is chosen to be very large.

We can prove a *strong transversality* result for symplectic manifolds with distribution G along compact 2-calibrated subvarieties.

Theorem 3. *Let $E_k \to (P, \Omega)$ and let (M, D) be a compact 2-calibrated submanifold of the symplectic manifold (P, Ω) and G a J-complex distribution extending D. Let us consider $\mathcal{S}^G$ a C^h-A.H. sequence of finite Whitney quasi-stratifications of $\mathcal{J}^r E_k$ ($h \geq 2$). Let δ be a positive constant. Then a constant $\eta > 0$ and a natural number k_0 exist such that for any C^{r+h}-A.H.(C) sequence τ_k of E_k, it is possible to find a C^{r+h}-A.H. sequence σ_k of E_k so that for any k bigger than k_0,*

(1) *$|\nabla^j(\tau_k - \sigma_k)|_{g_k} < \delta, j = 0, \ldots, r+h$ ($\tau_k - \sigma_k$ is C^{r+h}-A.H.(δ))*
(2) *$j^r\sigma_k$ is η-transversal along M to $\mathcal{S}^G$*

Proof. We need to slightly modify the proof of theorem 1.1 in [4].

Step 1: Show that $(\eta_a, \bar{\eta}_a)$-transversality along M of $j^r\tau_k$ to ${S^G}^a_k$, for all $a < b$, implies the existence of $\bar{\eta}_b > 0$ such that $j^r\tau_k$ is $\bar{\eta}_b$-transversal along M to ${S^G}^b_k$ in the points $\bar{\eta}_b$-close to its boundary.

In theorem 1.1 [4], it is shown that the quasi-stratification condition together with full uniform transversality can be used to show that $j^r\tau_k$ stays uniformly away from ${S^G}^a_k \backslash \Theta_{{S^G}^a_k}$, say at distance greater than some $\eta' > 0$; since uniform transversality along M is stronger than uniform transversality we deduce the same result.

We now make use of the estimated Whitney condition as in corollary 6. We have the inequality

$$\angle_{\mathrm{m}}(T_M j^r\tau_k, T^{||}_M {S^G}^a_k) \leq \angle_{\mathrm{M}}(T^{||}_M {S^G}^a_k, T_M {S^G}^b_k) + \angle_{\mathrm{m}}(T_M j^r\tau_k, T_M {S^G}^b_k) \tag{50}$$

For $\eta'' > 0$ small enough the induction hypothesis implies that for points η''-close to $\bar{\partial}{S^G}^b_k$, there is some index $a \in A_k$ such that

$$\angle_{\mathrm{m}}(T_M j^r\tau_k, T^{||}_M {S^G}^a_k) \geq \eta_a$$

In order to make $\angle_{\mathrm{M}}(T^{||}_M {S^G}^a_k, T_M {S^G}^b_k) < \eta_a/2$, we use the estimated Whitney condition that gives $\angle_{\mathrm{m}}(\hat{M}, T{S^G}^b_k) > \gamma$ and $\angle_{\mathrm{M}}(T^{||}{S^G}^a_k, T{S^G}^b_k) < C(\gamma)^{-1}\eta_a/2$ (see the proof of corollary 6), for η'' small enough. Then the desired result holds for $\bar{\eta}_b := \min(\eta', \eta'', \min_{a<b}(\eta_a/2))$.

Step 2: Let $\eta, \bar{\eta} > 0$ be constants. A C^{r+2}-A.H. sequence of sections τ_k of E_k is said to verify $\mathcal{P}_k(\eta, \bar{\eta}, x)$ if $j^r\tau_k$ is $(\eta, \bar{\eta})$-transversal along M to ${S^G}^b_k$ at x. We need to show that this is a C^{r+2}-open condition, i.e that for χ_k with $|\tau_k - \chi_k|_{C^{r+2}, g_k} \leq \epsilon$, if τ_k verifies $\mathcal{P}_k(\eta, \bar{\eta}, x)$ then χ_k verifies $\mathcal{P}_k(\eta - L\epsilon, \bar{\eta} - L\epsilon, x)$, where $L > 0$ is independent of k, x (and does not depend either on τ_k but on its C^{r+2}-bounds).

Again this is done in theorem 1.1 [4] for full transversality. For estimated transversality along M is equally true because a perturbation χ_k with C^{r+2}-size bounded by C, gives rise to an r-jet such that (i) $|j^r\chi_k|_{g_k} \leq L'C$, (ii) $|\nabla_{TM} j^r\chi_k|_{g_k} \leq L'C$

and (iii)$|\nabla\nabla_{TM} j^r\chi_k|_{g_k} \leq L'C$, for some $L'>0$. Therefore, small perturbations of a given section give rise to an r-jet that remains within controlled distance of the one for the initial section and whose derivative along TM varies in a controlled way. Similarly, for a given r-jet we can control in a ball of uniform radius its variation up to order 2, and hence the variation of its derivative along TM in the ball.

Step 3: To deal with points far from the boundary, we fix coordinates adapted to M and use instead of the local transversality result in theorem 1.1. [4] (proposition 4.2), the following local transversality result, which is a reformulation of lemma 5.2 and theorem 5.4 in [27].

Proposition 8. *Let F be a function with values in $\mathbb{C}^l$ defined over the ball of radius 11/10 in $\mathbb{C}^l$. Let V a vector subspace of $\mathbb{C}^l$. Let δ be a constant $0<\delta<1/2$. Let $\eta=\delta(P(\log(\delta^{-1}))^{-1}$, where P is a real monomial depending on n,l,V. If in the ball of radius 11/10 we have*

$$|F|_{g_0}\leq 1,\ |\bar\partial F|_{g_0}\leq\eta,\ |\mathrm{d}\bar\partial F|_{g_0}\leq\eta,$$

then there exists $u\in\mathbb{C}^p$ such that $F-u$ is η-transversal to $\mathbf{0}$ along V in the interior (in V) of $B(0,1)\cap V$.

Step 4: Substitute proposition 4.1 in [4] by an analogous one in which from local perturbations using linear combinations of reference frames that give $\mathcal{P}_k(C\delta(P(\log(\delta^{-1}))^{-1},\delta,y)$, $y\in B_{g_k}(x,c)$, we obtain $\mathcal{P}_k(\epsilon,\bar\epsilon,x)$, for all $x\in M$ (this is done in section 5.5 in [27]).

Recall that the proof of theorem 1.1. in [4] goes roughly as follows: we assume by step 1 that τ_k is already $\bar\eta_b$-transversal along M in the points $\bar\eta_b$-close to the boundary. Let $x\in M\subset P$ and $0<\epsilon<\bar\eta_b/4$. If ϵ is small enough and $j^r\tau_k(x)=p\in\mathcal{N}_{SG^b_k}(\epsilon/2,\bar\eta_b)$, then there exists ρ (all our constants always independent of k,x and strictly positive) such that $j^r\tau_k(B_{g_k}(x,\rho_1))\subset B_{\hat g_k}(p,\rho_\epsilon)\subset\mathcal{N}_{SG^b_k}(\epsilon,3\bar\eta_b/4)$. We consider the composition $f\circ j^r\tau_k$ pulled back to the domain of an A.H. chart adapted to M and centered at x. In this way we obtain a function $H_k\colon B(0,\rho_2)\subset\mathbb{C}^p\to\mathbb{C}^l$. If we apply proposition proposition 8 directly to H_k and for $\delta\ll\bar\eta_b/6$, we will obtain $\delta(P(\log(\delta^{-1})))^{-1}$-transversality to $\mathbf{0}$ along M for H_k-u_k in $B_{g_k}(x,\rho_3)$. The problem is how to associate u_k to a perturbation of τ_k (the difficulty coming from the non-linearity of the strata). Instead, we consider for each index I the $\mathbb{C}^l$-valued function such that for each $y\in B_{g_k}(x,\rho_4)$

$$\Theta_I(y)=(df_1(j^r\tau_k(y))j^r\tau^{\mathrm{ref}}_{k,x,I},\dots,df_l(j^r\tau_k(y))j^r\tau^{\mathrm{ref}}_{k,x,I}),$$

with $\tau^{\mathrm{ref}}_{k,x,I}$ as defined in equation 25. There is a choice of l indices $I_1,\dots,I_l$ such that the corresponding A.H. sections $j^r\tau^{\mathrm{ref}}_{k,x,I_j}$ are a frame for a distribution complementary to Kerdf (and with minimal angle bounded from below). Then $\Theta_{I_1},\dots,\Theta_{I_l}$ is a frame (depending on y) of $\mathbb{C}^l$ comparable to the canonical one. We can write

$$H_k=h^1_k\Theta_{I_1}+\cdots+h^l_k\Theta_{I_l}$$

We apply proposition 8 (after suitable rescalings) to the $\mathbb{C}^l$-valued function $h_k=(h^1_k,\dots,h^l_k)$, for some δ small enough, so we get $u_k\in\mathbb{C}^l$ so that h_k-u_k is $c_1\delta(P(\log(\delta^{-1})))^{-1}$-transversal along M to $\mathbf{0}$ in $B_{g_k}(x,\rho_5)$. If we multiply by the functions $\Theta_{I_1},\dots,\Theta_{I_l}$ we obtain $c_2\delta(P(\log(\delta^{-1})))^{-1}$-transversality along M to $\mathbf{0}$ for $H_k-u^1_k\Theta_{I_1}-\cdots-u^l_k\Theta_{I_l}$.

We consider the section $s_{k,x}=-u^1_k\tau^{\mathrm{ref}}_{k,x,I_1}-\cdots-u^l_k\tau^{\mathrm{ref}}_{k,x,I_l}$.

The key point is that having into account the norm of u_k and the bounds on the second derivatives of f, the C^1-norm of

$$H_k-u^1_k\Theta_{I_1}-\cdots-u^l_k\Theta_{I_l}-f\circ j^r(\tau_k+s_{k,x})$$

is bounded by $O(\delta^2)$. Since the C^1-norm majorates the C^1-norm along M, we conclude that for δ small enough, $f \circ j^r(\tau_k + s_{k,x})$ is $c_3\delta(P(\log(\delta^{-1})))^{-1}$-transversal along M to $\mathbf{0}$. By lemma 11 we get $\mathcal{P}_k(c_4\delta(P(\log(\delta^{-1})))^{-1}, \bar{\eta}_b - L\delta, y)$, for all $y \in B_{g_k}(x, \rho_5)$. Notice that by step 2 $\mathcal{P}_k(\eta, \bar{\eta}, x)$ is C^{r+2}-open, so if δ is small enough compared to $\bar{\eta}_b$ and $\eta_a, \bar{\eta}_a$, we still get uniform transversality to the previous strata and $5\bar{\eta}_b/6$-transversality along M in the points $3\bar{\eta}/4$-close to the boundary of $S_k^{G^b}$.

So we can apply step 4 to obtain $\mathcal{P}_k(\eta_b, 3\bar{\eta}_b/4, x)$ (w.r.t. $S_k^{G^b}$) in all the points of M.

Hence we deduce the existence of a C^{r+2}-A.H. sequence σ_k such that:

(1) $|\nabla^j(\tau_k - \sigma_k)|_{g_k} < \delta, j = 0, \ldots, r+h$ (σ_k is C^{r+2}-A.H.(δ))
(2) $j^r\sigma_k$ is η-transversal to $\mathcal{S}^G$ along M

□

8. Applications

We begin by proving proposition 1, which can be also obtained as a simple corollary of the work of J.-P. Mohsen [27] together with some extra local work borrowed from [24].

Proof of proposition 1. We consider a more general situation than that of the statement of proposition 1. Let E be any rank m hermitian vector bundle over (M^{2n+1}, D, ω), and let $E_k = E \otimes L_\Omega^{\otimes k}$ (L_Ω the pre-quantum line bundle of the symplectization and E is meant to be the pullback of the initial E to the symplectization). We want to apply theorem 3 to the sequence of zero sections Z_k, but with some changes. We fix A.H. coordinates adapted to M and reference sections $\tau^{\mathrm{ref}}_{k,x,j}$ centered at the points of $M \subset M \times [-\epsilon, \epsilon]$. The uniformly transversal section will be obtained by suitably adding local perturbations. In A.H. coordinates adapted to M we take the sections $z_k^j \tau^{\mathrm{ref}}_{k,y,j}$, $j = 1, \ldots, m \leq n+1$ and consider its direct sum, a section of E_k. This sequence of sections $\tau_{k,y}$ vanishes on y and it η-transversal along M to Z_k in $B_{g_k}(y, \rho)$. The point is to keep on adding local perturbations (centered at other points) which vanish at y, and with C^1-norm small enough compared to η. For that we need new reference sections vanishing at y. Notice that if $d_k(x, y) \geq O(k^{1/6})$, then $\tau^{\mathrm{ref}}_{k,x,j}$ is already vanishing at y, so we do not need to change the reference section. Assuming $d_k(x, y) \leq O(k^{1/6})$, once we go to A.H. coordinates adapted to M and centered at x the point y belongs to $B(0, O(k^{1/6})) \subset \mathbb{C}^{n+1}$. Consider the polynomial $P(z_k^1, \ldots, z_k^{n+1}) = 1 - z_k^1$. Let $L_{k,y,x} \in Gl(n+1, \mathbb{C})$ be the composition of dilatation and then a rotation sending y to $(1, 0, \ldots, 0)$. We define $P_{k,y,x} = P \circ L_{k,y,x}$ and $\xi^{\mathrm{ref}}_{k,x,j} := P_{k,y,x}\tau^{\mathrm{ref}}_{k,x,j}$. For any $\gamma > 0$, if we suppose $d_k(x, y) \geq \gamma$ then $\xi^{\mathrm{ref}}_{k,x,j}$ becomes an A.H. sequence (with bounds independent of x) that vanishes at y and so that $\xi^{\mathrm{ref}}_{k,x,j}$, $j = 1, \ldots, m$ fits into a local frame of E_k over $B_{g_k}(x, \rho(\gamma))$. Since $\tau_{k,y}$ is η-transversal along M to Z_k in $B_{g_k}(y, \rho)$, we only need to add perturbations centered at points away from $B_{g_k}(y, \rho/2)$, and thus the globalization procedure can be applied with reference sections vanishing at y.

Thus it is possible to find sequences of A.H. sections τ_k of E_k uniformly transversal along M to Z_k and vanishing at y. Hence $\tau_{k|M}$ are uniformly transversal to Z_k and vanishing at y. Let $W_k = \tau_{k|M}^{-1}(Z_k)$. For all $k \gg 1$, by corollary 5 they are uniformly transversal to D and by lemma 18 approximately almost complex and therefore 2-calibrated.

The study its topology is done very much as in the symplectic and contact cases (see the proofs in [9, 2, 20]). □

The next result we want to prove the existence of determinantal submanifolds (proposition 2), that is still a transversality result for 0-jets (vector bundles E_k), but not anymore to the $\mathbf{0}$ section but to a sequence of non-linear approximately holomorphic stratifications.

Proof of proposition 2. Let $E, F \to M$ be hermitian bundles with connection and let us define the sequence of very ample vector bundles $I_k := E^* \otimes F \otimes L^{\otimes k}$. In the total space of I_k we consider the sequence of stratifications S_k whose strata are $S_{k,i} = \{A \in I_k |\ \mathrm{rank}(A) = i\}$, where $A \in \mathrm{Hom}(E, F \otimes L^{\otimes k})$.

Let E, F still denote the pullback of E, F to the symplectization. Let $I_{k,\Omega} \to M \times [-\epsilon, \epsilon]$ be $E^* \otimes F \otimes L_\Omega^{\otimes k}$. Let G as usual a J-complex distribution defined on $M \times [-\epsilon, \epsilon]$ that extends D, and let

$S^G_{k,i} = \{A \in I_{k,\Omega} |\ \mathrm{rank}(A) = i\}$, where $A \in \mathrm{Hom}(E, F \otimes L_\Omega^{\otimes k})$.

By lemma 13 (applied to almost complex manifolds) $S^G_{k,i}$ is an approximately holomorphic sequence of finite Whitney stratifications. Therefore we can apply theorem 3 to construct an A.H. sequence of sections τ_k of $I_{k,\Omega}$ uniformly transversal to S^G_k along M and therfore along D.

Hence for all k large enough M is stratified by the submanifolds $S_i(\tau_k) = \{x \in M |\ \mathrm{rank}(\tau_k(x)) = i\}$, which are uniformly transversal to D and 2-calibrated by lemma 18. □

Corollary 1 follows from the fact that in the contact case the 2-form is exact and hence the cohomological computations are those of the bundle $E^* \otimes F$.

Theorem 4. *Let (M, D, ω) be an integral 2-calibrated manifold and set $E_k = \underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$ and let r any natural number. Any A.H. sequence of sections of $\underline{\mathbb{C}}^{m+1} \otimes L_\Omega^{\otimes k} \to (M \times [-\epsilon, \epsilon], \Omega, G)$ admits an arbitrarily small C^{r+h}-perturbation such that $\phi_{k|M} \colon M \backslash B_k \to \mathbb{CP}^m$ -the restriction to M of its projectivization- is an r-generic A.H. sequence.*

Proof. The proof is just theorem 3 applied to the Thom-Boardman-Auroux quasi-stratification along G of $\mathcal{J}^r E_k \to (M \times [-\epsilon, \epsilon], J, G, g_k)$ combined with proposition 7. □

It must be pointed out that the behavior of A.H. functions in the points close to the degeneration loci is more complicated than that of the leafwise holomorphic model: firstly, and similarly to what happens for even dimensional a.c. manifolds, to obtain normal forms it is necessary to add perturbations so that the function becomes holomorphic (at least in certain directions); otherwise the approximate holomorphicity is not significative due to the vanishing (degeneracy) of the holomorphic part. Secondly, we have an extra non-holomorphic direction that we do not control. At most, we can apply the usual genericity results to that direction (but perturbations of size $O(k^{-1/2})$ so as not to destroy the other properties).

One instance of the preceding theorem is when the target space has large dimension so that the generic map is an immersion along the directions of D.

Proof of corollary 2. Set $E_k = \underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$, where $m \geq 2n$.

Theorem 3 is applied to the Thom-Boardman-Auroux quasi-stratification along G of $\mathcal{J}^1 E_k \to (M \times [-\epsilon, \epsilon], J, G, g_k))$, to obtain 1-generic A.H. maps $\phi_k \colon M \to \mathbb{CP}^m$. From the choice of m it follows that the set of base points and of points where $\partial\phi_k$ is not injective is empty. It is clear that by construction $\phi_k^*[\omega_{FS}] = [\omega_k]$. □

This is a non-trivial result because the property of being an immersion along D is not generic (for smooth maps to $\mathbb{CP}^{2n}$). Notice that if for example D is integrable the property is generic for each leaf (locally), but not for the 1-parameter family.

As mentioned in the introduction, the previous corollary can be improved in two different ways.

Proof of corollary 3. Let us assume that any 2-form in the path $\rho_{k,t} = (1-t)\omega_k + t\phi_k^*\omega_{FS}$ is non-degenerate over $\mathcal{D}$, where ω_{FS} is be the Fubini-Study 2-form. Then the Moser trick can be applied leafwise: if α is a 1-form such that $d\alpha = -(\phi_k^*\omega_{FS} - \omega_k)$, the vector fields tangent to $\mathcal{D}$ defined by the condition $-i_{X_t}\rho_{k,t} = -\alpha$ generate a 1-parameter family of diffeomorphisms preserving each leaf and sending $\rho_{k,t}$ to ω_k.

The non-degeneracy over $\mathcal{D}$ of ρ_t follows from the estimated transversality of ϕ_k together with the approximate holomorphicity. For any $v \in D_x$ of g_k-norm 1,

$$\rho_{k,t}(v, Jv) = (1-t)\omega_k(v, Jv) + t\omega_{FS}(\phi_{k*}v, \phi_{k*}Jv) \geq (1-t) + t\eta > 0$$

□

In general a closed Poisson manifold with codimension 1 leaves does not admit a lift to a 2-calibrated structure (for example any non-taut smooth foliation in M^3). The previous corollary can be used to state the following result:

Corollary 7. *Let $(M^{2n+1}, \mathcal{D}, \omega_{\mathcal{D}})$ be a closed Poisson manifold with co-oriented codimension 1 leaves. Then the Poisson structure admits a lift to a (rational) 2-calibrated structure if and only if a multiple of $\omega_{\mathcal{D}}$ is induced by a leafwise immersion in $\mathbb{CP}^{2n}$ (by pulling back ω_{FS}).*

We finish this section by mentioning that it is possible to obtain uniform transversality to a finite number of quasi-stratifications of the same sequences of bundles. For example, and this leads to the second improvement of corollary 2, we can obtain the 1-genericity result that gives rise to embeddings in $\mathbb{CP}^m$ transversal to a finite number of complex submanifolds of $\mathbb{CP}^m$.

We just need to consider for each submanifold the sequence of stratifications $\mathbb{P}\mathcal{S}$ of $\mathcal{J}^1_G(M, \mathbb{CP}^m)$ whose unique stratum (for each k) is defined to be the 1-jets along G whose degree 0 component is a point of the submanifold; next we pull it back to a stratification $\mathcal{S}$ of $\mathcal{J}^1_G E_k^*$ and finally to a stratification $\mathcal{S}^G$ of $\mathcal{J}^1 E_k^*$ (the structure near Z_k is not relevant because transversality to the Thom-Boardman-Auroux quasi-stratification along G implies that the sections stay away from Z_k). Therefore, we have defined a stratification of $\mathcal{J}^1 E_k$ which is trivially approximately holomorphic because is the pullback by A.H. maps of an initial approximately holomorphic stratification of $\mathcal{J}^0_G(M, \mathbb{CP}^m)$.

Any 1-generic sequence of A.H. sections of E_k uniformly transversal to $\mathcal{S}^G$ when restricted to M gives rise to maps $\phi_k\colon M \hookrightarrow \mathbb{CP}^m$ uniformly transversal along D to the submanifold.

Proof of Theorem 1. We first apply theorem 4 to obtain $\phi_{k|M}\colon M\backslash B_k \to \mathbb{CP}^1$ 1-generic.

Near the base points and the points where $\nabla_D\phi_{k|M}$ vanishes, we apply the perturbations defined in [32] to obtain the required local models.

□

Another possible application is, as proposed by D. Auroux for symplectic manifolds [3, 4], to obtain r-generic applications to $\mathbb{CP}^m$ whose composition with certain projections $\mathbb{CP}^m \to \mathbb{CP}^{m-h}$ are still r-generic (the corresponding stratifications are approximately holomorphic because they are pullback of approximately holomorphic stratifications by A.H. maps; the structure near Z_k is also seen to be appropriate).

It is also possible to develop an analogous construction but for A.H. maps to grassmanians $Gr(r,m)$, starting from sections of $\underline{\mathbb{C}}^r \otimes E_k$, E_k of rank m (see [28, 5]).

Our techniques can be applied to any closed 2-calibrated manifold to give a finer topological description of the 2-calibrated structure. It is possible to apply the same idea to manifolds for which the 2-calibrated structure enters as an auxiliary tool. This point of view has already been adopted in [24].

We recall the following result.

Theorem 5. *(Gromov) Let M^{2n+1} a closed manifold whose structural group reduces to $U(n)$, and let $a \in H^2(M;\mathbb{Z})$. Then there exists ω a closed non-degenerate 2-form such that $[\omega]=a$.*

Proof. It is the result of applying the h-principle to the open manifold $M \times \mathbb{R}$, and then restricting the symplectic form to M. □

So by selecting any codimension 1 distribution transversal to the kernel of ω, we have:

Corollary 8. *Let M^{2n+1} a closed manifold whose structural group reduces to $U(n)$, and let $a \in H^2(M;\mathbb{Z})$. Then M admits 2-calibrated structures (D,ω) for which $[\omega]=a$.*

Notice that if we apply any of the previous constructions to (M,D,w), we obtain submanifolds and more generally stratifications of M by 2-calibrated submanifolds. Regarding the initial structure, which was just a reduction of the structural group to $U(n)$, we can conclude that the corresponding strata also admit such a reduction.

9. Proof of proposition 4

We write down the proof for the bundle $\mathcal{J}^r E_k$, for it is a necessary ingredient in the proof of theorem 3. The case of $\mathcal{J}^r_D E_k$ bears no further complications and it is left to the interested reader.

We omit the subindexes k and r for the connections whenever there is no risk of confusion.

Recall that in coordinates the curvature can be computed as follows: in a chart where T^*P is trivialized using the derivatives of the coordinates, we have the corresponding flat connection d on T^*P. We have the operator

$$\begin{array}{rcl} \nabla^1 \colon T^*P \otimes L_k & \longrightarrow & T^*P \otimes T^*P \otimes L_k \\ \nabla^1 & := & \mathrm{d} \otimes \mathrm{I} - \mathrm{I} \otimes \nabla \end{array}$$

and the antisymmetrization map

$$\begin{array}{rcl} \mathrm{asym}_2 \colon T^*P \otimes T^*P & \longrightarrow & \wedge^2 T^*P \\ \alpha \otimes \beta & \longmapsto & \alpha \wedge \beta \\ \alpha \wedge \beta(u,v) := \alpha(u)\beta(v) - \alpha(v)\beta(u) & & \end{array}$$

The curvature is the composition $\mathrm{asym}_2(\nabla^1 \circ \nabla)$.

Let $\sigma_k = (\sigma_{k,0}, \sigma_{k,1})$ be a section (maybe local) of $\mathcal{J}^1 E_k$. The modified connection is $\nabla_{H_1}(\sigma_{k,0}, \sigma_{k,1}) = (\nabla \sigma_{k,0}, \nabla \sigma_{k,1}) + (0, -F^{1,1}\sigma_{k,0})$, where $-F^{1,1}\sigma_{k,0} \in T^{*0,1}P \otimes T^{*1,0}P \otimes E_k$ (see [5]). For jets along D we add $-F^{1,1}_D$.

The previous formula defines a connection.

Lemma 19. *Let $\underline{\mathbb{C}}^m \to \mathbb{C}^p$ be the trivial bundle endowed with a connection ∇ whose curvature is of type $(1,1)$ with respect to the canonical complex structure J_0;*

the connection splits into $\partial_\nabla + \bar{\partial}_\nabla$*. Let* τ *be a holomorphic section of* $\underline{\mathbb{C}}^m$ *(w.r.t. to the holomorphic structure induced by* ∇*). Then*

$$\nabla_H(\tau, \partial_\nabla \tau) = \nabla(\tau, \partial_\nabla \tau) - (0, \bar{\partial}_\nabla \partial_\nabla \tau) \tag{51}$$

and $\bar{\partial}_{\nabla_H}(\tau, \partial_\nabla \tau) = 0$.

Proof. By definition

$$F\tau = \mathrm{asym}_2(\nabla^1 \nabla \tau) \tag{52}$$

Let us denote the trivialization of the bundle that identifies it with $\underline{\mathbb{C}}^m$ by $\xi_1, \ldots, \xi_m$.

Since τ is holomorphic,

$$F\tau = \mathrm{asym}_2((\mathrm{d} \otimes \mathrm{I} - \mathrm{I} \otimes \nabla)\partial_\nabla \tau) \tag{53}$$

If we write $\partial_\nabla \tau = dz^i h_i^j \xi_j$, then

$$F\tau = \mathrm{asym}_2(-(\mathrm{I} \otimes \nabla) dz^i h_i^j \xi_j) \tag{54}$$

Being the curvature of type $(1,1)$,

$$F\tau = \mathrm{asym}_2(-(\mathrm{I} \otimes \bar{\partial}_\nabla) dz^i h_i^j \xi_j) \tag{55}$$

Recall that $F\tau$ has to be understood as an element of $T^{*0,1}\mathbb{C}^p \otimes T^{*1,0}\mathbb{C}^p \otimes \underline{\mathbb{C}}^m$. That amounts to switch the $d\bar{z}^l$'s with the dz^l's, which cancels the negative sign on the r.h.s. of equation 55. Thus what we obtain is:

$$F\tau := (\mathrm{I} \otimes \bar{\partial}_\nabla) dz^i h_i^j \xi_j \in \Gamma(T^{*0,1}\mathbb{C}^p \otimes T^{*1,0}\mathbb{C}^p \otimes \underline{\mathbb{C}}^m) \tag{56}$$

But equation 55 equals

$$(\bar{\partial}_0 \otimes \mathrm{I} + \mathrm{I} \otimes \bar{\partial}_\nabla) dz^i h_i^j \xi_j$$

which by definition is

$$\bar{\partial}_\nabla \partial_\nabla \tau \tag{57}$$

By equation 57

$$\bar{\partial}_{\nabla_H}(\tau, \partial_\nabla \tau) = (\bar{\partial}_\nabla \tau, \bar{\partial}_\nabla \partial_\nabla \tau - \bar{\partial}_\nabla \partial_\nabla \tau) = 0$$

□

It is also clear that $\partial_\nabla = \partial_{\nabla_H}$, and therefore they define the same coupled holomorphic jets.

Lemma 19 has an obvious approximately holomorphic version: if we have a very ample sequence of rank m vector bundles by definition the sequences of curvatures is approximately of type $(1,1)$. Then we can fix approximately holomorphic coordinates and the first part of lemma 19 implies that for τ_k a sequence of A.H. sections of E_k, one has

$$F\tau_k \approxeq \bar{\partial}\partial \tau_k,$$

and by the second part

$$\bar{\partial}_H j^1 \tau_k \approxeq 0,$$

We compute the curvature of the modified connection in the integrable case. We will denote the coupled holomorphic r-jet in the integrable model by $j^r_{\mathrm{hol}}\tau$.

Lemma 20. *Let $\underline{\mathbb{C}}^m \to \mathbb{C}^p$ be the trivial bundle as in lemma 19. Assume also that for the fixed trivialization $\xi_1, \ldots, \xi_m$ the curvature is a matrix with constant coefficients and that we have a frame given by holomorphic sections $\tau_1, \ldots, \tau_m$. Then $F_\nabla = F_{\nabla_H}$*

Proof. If the holomorphic sections $\tau_1, \ldots, \tau_m$ generate the bundle, then the holomorphic 1-jets of $z^l \tau_j$, τ_j, $1 \leq l \leq p$, $1 \leq j \leq m$ are a basis of $\mathcal{J}^1_{p,m}$ (at least on $B(0, O(1))$). By lemma 19, they are a holomorphic basis.

$$\nabla_H j^1_{\mathrm{hol}} z^l \tau_j = (\partial_\nabla(z^l\tau_j), \nabla\partial_\nabla(z^l\tau_j)) - (0, Fz^l\tau_j) = \nabla j^1_{\mathrm{hol}} z^l\tau_j - (0, Fz^l\tau_j) \quad (58)$$

Let us write again $\partial_\nabla \tau_j = dz^i h^s_{i,j}\xi_s$, and $F = a_{ts} d\bar{z}^t dz^s \in \Gamma(T^{*0,1}\mathbb{C}^p \otimes T^{*1,0}\mathbb{C}^p)$. If we apply to $\nabla j^1_{\mathrm{hol}} z^l\tau_j$ the operator $\mathrm{asym}_2 \nabla^1_H$, $\nabla^1_H := \mathrm{d} \otimes \mathrm{I} - \mathrm{I} \otimes \nabla_H$, we get:

$$F_\nabla j^1_{\mathrm{hol}} z^l\tau_j + (0, \mathrm{asym}_2(dz^l a_{ts} d\bar{z}^t dz^s \tau_j + z^l dz^i a_{ts} d\bar{z}^t dz^s h^s_{i,j}\xi_s)) \quad (59)$$

When we apply the same operator to $(0, Fz^l\tau_j)$, if recall that the a_{ts} are constant and that $z^l\tau_j$ is a holomorphic section,

$$\mathrm{asym}_2 \nabla^1_H (0, Fz^l\tau_j) = (0, \mathrm{asym}_2(-a_{ts} d\bar{z}^t dz^l dz^s \tau_j - a_{ts} d\bar{z}^t z^l dz^i h^s_{i,j}\xi_s)) \quad (60)$$

and the r.h.s. of equation 60 equals

$$(0, \mathrm{asym}_2(dz^l F\tau_j + z^l dz^i F h^s_{i,j}\xi_s)) \quad (61)$$

If we put together equations 58, 59 and 61 we obtain

$$F_{\nabla_H}\tau_j = F_\nabla \tau_j$$

□

We want to use a recursive construction based on lemmas 19 and 20 to introduce the desired connection on $\mathcal{J}^r_{p,m}$.

Before doing that we recall that the coupled holomorphic jets are sections of $\mathcal{J}^r_{p,m}$.

We now proof how to modify the connection on $\mathcal{J}^2_{p,m}$.

Step 1: We identify $\mathcal{J}^2_{p,m}$ with the subbundle of $\mathcal{J}^1\mathcal{J}^1_{p,m}$ spanned by holonomic sections, i.e. sections of the form $j^1_{\mathrm{hol}} j^1_{\mathrm{hol}}\tau$, where τ is any holomorphic section of $\underline{\mathbb{C}}^m$.

Pointwise, an element γ of the fiber of $\mathcal{J}^1\mathcal{J}^1_{p,m}$ is of the form

$$(\gamma_{0,0}, \gamma_{0,1}, \gamma_{1,0}, \gamma_{1,1}) \in (\mathbb{C} \oplus T^{*1,0}\mathbb{C}^p \oplus T^{*1,0}\mathbb{C}^p \oplus (T^{*1,0}\mathbb{C}^p \otimes T^{*1,0}\mathbb{C}^p)) \otimes \mathbb{C}^m,$$

and belongs to $\mathcal{J}^2_{p,m}$ if and only if $\gamma_{1,1} \in T^{*1,0}\mathbb{C}^p \odot T^{*1,0}\mathbb{C}^p \otimes \mathbb{C}^m$ and $\gamma_{1,0} = \gamma_{0,1}$.

Using the metric induced by the Euclidean one on the base and fiber and the connection, we have a orthogonal projection $r\colon \mathcal{J}^1\mathcal{J}^1_{p,m} \to \mathcal{J}^2_{p,m}$.

Step 2: We introduce a new connection on $\mathcal{J}^1\mathcal{J}^1_{p,m}$.

On $\mathcal{J}^1_{p,m}$ we use the modified connection ∇_{H_1}. This, together with the flat connection d on $T^*\mathbb{C}^p$ defines a connection $\nabla_{H_{1,1}}$ on $\mathcal{J}^1\mathcal{J}^1_{p,m}$. Notice that on $\mathcal{J}^1\mathcal{J}^1_{p,m}$ we also have a connection ∇_2 coming from d and ∇_1.

We consider the trivialization of $\mathcal{J}^1_{p,m}$ furnished by the sections ξ_j, $dz^i\xi_j$, $1 \leq j \leq m$, $1 \leq i \leq p$, so we can identify the bundle with $\underline{\mathbb{C}}^{mp+m}$. This is a trivial bundle with connection ∇_{H_1}. By lemma 20 $F_{\nabla_{H_1}} = F_{\nabla_1}$. Recall also that in the basis ξ_j, $dz^i\xi_j$ the curvature F_{∇_1} is a matrix that decomposes into $p+1$ blocks corresponding to $\xi_1, \ldots, \xi_m$ and to $dz^i\xi_1, \ldots, dz^i\xi_m$, $1 \leq i \leq p$. For each such block

the corresponding matrix is the one for F_∇ in the basis ξ_j. Therefore $F_{\nabla_{H_1}}$ is still of type $(1,1)$ and has constant entries in the aforementioned basis.

Let ∇_{H_2} be the result of modifying $\nabla_{H_{1,1}}$.

Since ∇_{H_1} is of type $(1,1)$ by lemma 19 applied to $(\underline{\mathbb{C}}^{mp+m}, \nabla_{H_1})$, if $\tau^1 \in \Gamma(\mathcal{J}^1_{p,m})$ is holomorphic w.r.t. ∇_{H_1}, then $j^1_{\mathrm{hol}}\tau^1$ is holomorphic w.r.t. ∇_{H_2}. In particular $j^1_{\mathrm{hol}}(j^1_{\mathrm{hol}}z^i\tau_j)$, $j^1_{\mathrm{hol}}(z^l j^1_{\mathrm{hol}}z^i\tau_j)$ are a local holomorphic frame of $(\mathcal{J}^1\mathcal{J}^1_{p,m}, \nabla_{H_2})$ (recall that τ_j was a local holomorphic frame of $\underline{\mathbb{C}}^m$).

Having into account that the curvature of $(\underline{\mathbb{C}}^{mp+m}, \nabla_{H_1})$ is of type $(1,1)$ and with constant entries, and that $(\underline{\mathbb{C}}^{mp+m}, \nabla_{H_1})$ has a local holomorphic basis, lemma 20 gives $F_{\nabla_{H_2}} = F_{\nabla_{H_{1,1}}}$. From $F_{\nabla_{H_1}} = F_{\nabla_1}$ it follows that $F_{\nabla_{H_{1,1}}} = F_{\nabla_2}$. Therefore,

$$F_{\nabla_{H_2}} = F_{\nabla_2} \text{ on } \mathcal{J}^1\mathcal{J}^1_{p,m} \tag{62}$$

Step 3: Check that ∇_{H_2} restricts to $\mathcal{J}^2_{p,m} \hookrightarrow \mathcal{J}^1\mathcal{J}^1_{p,m}$ with the desired properties.

Let $I = (i_0, i_1, \ldots, i_p)$ with $1 \leq i_0 \leq m$, $0 \leq i_j \leq 2$, $i_1 + \cdots + i_p \leq 2$, and let $\tau_I := z_1^{i_1} \ldots z_p^{i_p}\tau_{i_0}$.

We consider the sections $j^1_{\mathrm{hol}}j^1_{\mathrm{hol}}\tau_I$, which are a local holomorphic frame $\mathcal{J}^2_{p,m}$ (using the identification described in step 1). We will see that $\nabla_{H_2}j^1_{\mathrm{hol}}j^1_{\mathrm{hol}}\tau_I \in \Gamma(T^{*1,0}\mathbb{C}^p \otimes \mathcal{J}^2_{p,m})$, and therefore that the connection ∇_{H_2} preserves $\mathcal{J}^2_{p,m}$.

We just proved in step 2 that $j^1_{\mathrm{hol}}j^1_{\mathrm{hol}}\tau_I$ is holomorphic w.r.t ∇_{H_2} and that $\partial_{\nabla_{H_2}} = \partial_{\nabla_{H_{1,1}}} = \partial_{\nabla_2}$. Let us write $j^1_{\mathrm{hol}}j^1_{\mathrm{hol}}\tau_I = (\tau_I, \partial_\nabla\tau_I, \partial_\nabla\tau_I, \partial^2_\nabla\tau_I)$. Then:

$$\begin{aligned} \nabla_{H_2}j^1_{\mathrm{hol}}j^1_{\mathrm{hol}}\tau_I &= \partial_{\nabla_{H_2}}j^1_{\mathrm{hol}}j^1_{\mathrm{hol}}\tau_I = \\ \partial_{\nabla_2}(\tau_I, \partial_\nabla\tau_I, \partial_\nabla\tau_I, \partial^2_\nabla\tau_I) &= (\partial_\nabla\tau_I, \partial_\nabla\partial_\nabla\tau_I, \partial_\nabla\partial_\nabla\tau_I, \partial_\nabla\partial^2_\nabla\tau_I), \end{aligned}$$

which belongs to $\Gamma(T^{*1,0}\mathbb{C}^p \otimes \mathcal{J}^2_{p,m})$.

Therefore, the curvature of the restriction of ∇_{H_2} to $\mathcal{J}^2_{p,m}$ is of course of type $(1,1)$. The last observation is its expression in a suitable basis. The curvature of ∇_2 on $\mathcal{J}^1\mathcal{J}^1_{p,m}$ splits on blocks corresponding to the basis $\xi_1, \ldots, \xi_m$, $dz^i\xi_1, \ldots, dz^i\xi_m$, $dz^l\xi_1, \ldots, dz^l\xi_m$, $dz^r \otimes dz^t\xi_1, \ldots, dz^r \otimes dz^t\xi_m$, $1 \leq i, l, r, t \leq p$. Each submatrix is F_∇. If we use the basis $\xi_1, \ldots, \xi_m$, $dz^i\xi_1, \ldots, dz^i\xi_m$, $dz^r \odot dz^t\xi_1, \ldots, dz^r \odot dz^t\xi_m$, $1 \leq i, r, t \leq p$ the curvature equally splits into blocks each matching F_∇.

The general case uses the following induction step: on $\mathcal{J}^r_{p,m}$ there exists a connection ∇_{H_r} with the following properties:

(1) $\partial_{H_r} = \partial_r$
(2) $F_{\nabla_{H_r}} = F_{\nabla_r}$ and therefore $F_{\nabla_{H_r}}$ is of type $(1,1)$.
(3) If $\bar\partial_\nabla\tau = 0$ then $\bar\partial_{H_r}j^r_{\mathrm{hol}}\tau = 0$.
(4) In the basis $\xi_I := (dz^1_k)^{\odot i_1} \cdots (dz^n_k)^{\odot i_n}\xi_{i_0}$ the curvature splits into blocks each matching F_∇.

To define $\nabla_{H_{r+1}}$ on $\mathcal{J}^{r+1}_{p,m}$ we reproduce the previous 3 steps.

Firstly we consider the identification of $\mathcal{J}^{r+1}_{p,m}$ with the subbundle of $\mathcal{J}^1\mathcal{J}^r_{p,m}$ spanned by sections of the form $j^1_{\mathrm{hol}}j^r_{\mathrm{hol}}\tau$, τ a holomorphic section of $\underline{\mathbb{C}}^m$.

Secondly we consider the connection $\nabla_{H_{1,r}}$ on $\mathcal{J}^{r+1}_{p,m}$ constructed out of d and ∇_{H_r} and modify it to $\nabla_{H_{r+1}}$.

By the induction hypothesis, using the basis ξ_I we are in the situation of lema 20, for $\mathcal{J}^r_{p,m}$ identifies with $\underline{\mathbb{C}}^{N_r}$ with a connection whose curvature is of type $(1,1)$ and with constant coefficients, and with a frame of holomorphic sections. Hence, $F_{\nabla_{H_{r+1}}} = F_{\nabla_{H_{1,r}}} = F_{\nabla_{r+1}}$.

Since we can also apply lemma 19, for any $\tau^r \in \Gamma(\mathcal{J}^r_{p,m})$ the 1-jet $j^1_{\mathrm{hol}}\tau^r$ is holomorphic w.r.t. $\nabla_{H_{r+1}}$.

The third step is to check that the modified connection restricts to $\mathcal{J}^{r+1}_{p,m} \hookrightarrow \mathcal{J}^1\mathcal{J}^r_{p,m}$. Using that $\partial_{\nabla_{H_{r+1}}} = \partial_{\nabla_{r+1}}$, any frame of sections of the form $j^1_{\mathrm{hol}} j^r_{\mathrm{hol}} \tau_I$, τ_I holomorphic, is sent by the connection to sections of $\mathcal{J}^{r+1}_{p,m}$.

It is also routine to check that in the basis ξ_I the curvature matrix is made of blocks of the form F_∇.

The almost complex counterpart of the result we just proved is done exactly in the same way. The only modification is that the connection on $\mathcal{J}^1\mathcal{J}^r E_k$ does not descend automatically to a connection on $\mathcal{J}^{r+1}E_k \hookrightarrow \mathcal{J}^1\mathcal{J}^r E_k$. We have to project via $r\colon \mathcal{J}^1\mathcal{J}^r E_k \to \mathcal{J}^{r+1}E_k$, but this is seen to introduce an error which is approximately vanishing. It might happen that the resulting connection amounts to adding also a pseudo-holomorphic part. If that is the case we forget about this contribution (which again would be approximately vanishing). Therefore, we obtain a connection with all the desired properties.

Using similar considerations to the ones for 1-jets, it can be deduced that the (r+1)-jet of a C^{r+1+h}-A.H. sequence of sections of E_k is a C^h-A.H. sequence of sections of $(\mathcal{J}^{r+1}E_k, \nabla_{H_{r+1}})$.

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